T. 5.1. alg.

$L/K$ ext of local fields.

A valuation $v$ on $K$ extends uniquely to $w$ on $L$ by

$$|g| = \sqrt{\frac{N_{L/K}(g)}{v}}.$$ 

$e(w/v) = [w(L^x): v(K^x)]$ ramification index

Have also $O_L, O_K$ valuation rings

$\mathfrak{m}_L, \mathfrak{m}_K$ max ideals

$\lambda = O_L/\mathfrak{m}_L, \kappa = O_K/\mathfrak{m}_K.$

$f(w/v) = [\lambda : \kappa]$ residue class degree.

Theorem. $e \cdot f = [L : K].$

(so, $L/K$ is unramified if $[L : K] = [\lambda : \kappa].$)
Def. $L/K$ (finite ext. of @p) is unramified if

$$[L : K] = [\lambda : k],$$

i.e. $e(L/K) = 1$.

An arbitrary algebraic extension $L/K$ is unramified if it is a union of finite unramified extensions.

Prop. (7.2) Given $L|k, k'|k$ inside a fixed alg. closure $\bar{F}$. Then,

$L|k$ unramified $\Rightarrow L \cdot k'|k'$ unramified.

Proof. Write $L' = L \cdot k'$

use the notation $\sigma, \varphi, k, \sigma', \varphi', k', \sigma, \varphi, \lambda, \sigma', \varphi', \lambda'$.

We argue just for finite extensions.

By the primitive element theorem $\lambda = k(\bar{a})$ for some $a \in \sigma$.

Write $f(x) \in \sigma[x]$ min poly of $a$, $\bar{f}(x) = f(x) \mod \varphi \in k[x]$.

Then

$$[\lambda : k] \leq \deg(\bar{f}) = \deg(f) = [k(\sigma) : k] \leq [L : K] = [\lambda : k]$$

so $L = k(\sigma)$ and $\bar{f}$ is the min poly of $\bar{a}$ over $k$.

So $L' = k'(\sigma)$.

Why is $L'|k'$ unramified?

Let $g(x) \in \sigma'[x]$ min poly of $a$ over $k'$.

$$g(x) = g(x) \mod \varphi' \in k'[x].$$

Note that $g(x)$ is a factor of $\bar{f}(x)$.

By Hensel’s Lemma $\bar{g}(x)$ is irreducible.

(If it factored, would lift to a factorization of $g(x)$.

So $[\lambda' : k'] \leq [L' : k] = \deg(g) = \deg(g) = [k'(\bar{a}) : k'] \leq [\lambda' : k'].$

So $[L' : k'] = [\lambda' : k']$, $L'|k'$ unramified.
If $L' | K$ is an unramified extension and $L \leq L'$, then $L | K$ is also unramified.

**Proof.** By prop. 1, $L' | L$ is unramified.

Have

$$[L' : K] = [\lambda_{L'} : \kappa]$$
$$[L' : L] = [\lambda_{L'} : \lambda_L].$$

Since field degrees are multiplicative, $L | K$ is ur.

(i.e. $[L : K] = [\lambda_L : \kappa]$.)

**Cor.** If $L$ and $L'$ are unramified over $K$, so is $LL'$.

**Proof.** $LL' | L'$ is unramified, with

$$[\lambda_{L'} : K] = [L' : K]$$
$$[\lambda_{LL'} : \lambda_{L'}] = [LL' : L'].$$

(Use: separability is transitive)

**Def.** Fix an algebraic closure $\overline{K}$ of $K$.

Then the composite of all unramified subextensions $L \subseteq \overline{K}$ of $K$ is the **maximal unramified extension** of $K$.

**Prop.** (7.5) The residue class field of $T$ is $\overline{E} (= \overline{Fp})$.

Moreover, $v(T^x) = v(K^x)$.

**Proof.** See Neukirch, but this is not hard.

(Tame ramification: 7.6, 7.7, 7.8, 7.9, 7.10, 7.11)
5.3. Def. Let $L/K$ be a finite extension of local fields.

Then, $L/K$ is tamely ramified if $\lambda | K$ is separable
(automatic for exts of $\mathbb{Q}_p$), and there exists an
intermediate field $T$ with

$T \mid K$ unramified

$[L : T]$ coprime to $p \ (= \text{char } K = \text{char } \lambda)$.  
(Typically $T = \text{max } \mathbb{Q}_{p^n}$ ext of $L/K$.)

1) cop. to $p$. Note: $\mathbb{Q}_p$ extensions are tamely ramified.

2) ur. Prop. 7.7. $L/K$ is tamely ramified iff $L \mid T$

is generated by radicals

$L = T(\sqrt[1]{\alpha_1}, \ldots, \sqrt[n]{\alpha_r})$ with $(\alpha_i, p) = 1$.

"Tame" = not too bad.

Go to the Jones - Roberts database.

Look at deg $n$ exts of $\mathbb{Q}_p$ when $p|n$. Then pln.

especially for $n$ also prime.

Very compelling...

Extensions of valuations.  (N. 2. 8.)

Bring back the global fields.

A local field has one valuation

A global field has a lot.

Interested in completions also.

(i.e. $\mathbb{Q}$ has the $p$-adic valuation

$\mathbb{Q}_p$ the completion.)
TS.4. Given $K$, number field.

valuation $v$, completion $K_v$, alg. closure $\bar{K}_v$.

Recall $v$ extends uniquely and canonically to $K_v$ (call it $v$ again)
and to $\bar{K}_v$ (call it $\bar{v}$).

Now, given $L/K_v$ we have an embedding

$\mathcal{L} \to \bar{K}_v$ fixing $K$.

(say a little bit...)

Restrict the valuation $\bar{v}$ to $\tau(L)$.

Label this valuation $w$. ($w = \bar{v} \circ \tau$.)

Can write this as $|x|_w = |\tau(x)|_{\bar{v}}$ for $x \in L$.

This map is continuous, and it extends uniquely to
a continuous $K$-embedding

$L_w \to \bar{K}_v$

$v = \omega \lim_{n \to \infty} x_n \to \tau x := \bar{v} \lim_{n \to \infty} \tau(x_n)$.

(Cauchy sequences map to Cauchy sequences.)

Have a field diagram

```
    Lw
   /
  /  \\
L   \L
    /
   /  \\
K
```

Canonical

Extension of $w$ from $L$ to $L_w$ is the unique extension
of $v$ from $K_v$ to $L_w$.
The extension $L_w$ satisfies $L_w = L K_v$. Why is this? $L K_v \leq L_w$ is again complete (4.8 - part we skipped proving).

Contains $L$ and so must be its completion.

As we saw before, $|x|_w = \sqrt[4]{|N_{L_w/L_v}(x)|_v}$.

This represents a local-to-global principle.

(Motivating example: $K = \mathbb{C}(t)$

$L = \text{alg. fns on some Riemann surface}$

Pass to $K_v$ and $L_w$: look at power series

(Take Jesse Kass's course)

Now the embedding $L \hookrightarrow \overline{K_v}$ was not necessarily unique.

There might be other such embeddings.

And, we got $w$ from $\tau$.

Example. Let $L/K = \mathbb{Q}(i)/\mathbb{Q}$.

There are two embeddings $\mathbb{Q}(i) \hookrightarrow \mathbb{Q}_5$.

How to find them? $2^2 \equiv -1 \pmod{5}$;

$3^2 \equiv -1 \pmod{5}$.

By Hensel's lemma, they lift uniquely.

Choose either for image of $i$.

There is no embedding $\mathbb{Q}(i) \hookrightarrow \mathbb{Q}_7$.

But there are two embeddings $\mathbb{Q}(i) \hookrightarrow \mathbb{Q}_7$.

Once we fix an algebraic closure of $\mathbb{Q}_7$ they are distinguished.

The image is $\mathbb{Q}_7[x]/(x^2 + 1)$ again write $\mathbb{Q}_7(i)$. 

Corollary. We have

\[ [L : K] = \sum_{w \mid v} [L_w : K_v] \]

\[ N_{L/K}(\sigma) = \prod_{w \mid v} N_{L_w/K_v}(\sigma) \quad \text{Tr}_{L/K}(\sigma) = \sum_{w \mid v} \text{Tr}_{L_w/K_v}(\sigma) \]

(i) is immediate.

(ii): on $L \otimes_K K_v = \prod_{w \mid v} L_w$

Look at the endomorphism multiplication by $\sigma$.

Char poly of $\sigma$ is the same on:

- $K$-vector space $L$
- $K_v$-vector space $L \otimes K_v$.

So char poly $L/K(\sigma) = \prod_{w \mid v}$ char poly $L_w/K_v(\sigma)$

and we get (ii).

efg for valuations.

Recall, for $w \mid v$, $e(w/v) = e_w = (w(L^v) : v(K^v))$

$f(w/v) = f_w = [L_w : K_v]$, 

and $[L_w : K_v] = e_w \cdot f_w$

(Prop 6.8: "efg" for local fields)

Therefore:

\[ \sum_{w \mid v} e(w/v) f(w/v) = [L : K] \]
Theorem 6.6. This is what we saw before.

Given $L/k$, $p$ in $K$, with $pO_L = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$,

$p \mapsto$ the $p$-adic valuation $v_p$ of $K$.

\[ v_p(a) = \# \text{ of } p \text{'s in the ideal factorization of } a \]

$p_i \mapsto$ the valuations $w_{p_i}$ extending $v_p$.

Check: the \{inertia degrees \} \{ramification indices\} match $v_p$

And so says the same thing as $\sum_{i=1}^r e_i f_i = [L:k]$. 
Theorem. Let $f(x)$ give an isomorphism

$L \otimes_K K_V \to \prod_{w|v} L_w$.

Proof. Let $L = K(a)$, and write (as before) $f(x) = \prod_{w|v} f_w(x)$

$f(x)$ min poly of $a$

Now, consider all the $L_w$ embedded in $K_V$,
write $t_w$: image of $a$ under $L \to L_w$, so that

$L_w = K_V(t_w)$, and

$f_w(x)$ is the min. polynomial of $a_w$ over $K_V$.

[Commutative diagram:]

\[
\begin{array}{ccc}
K_V[x]/(f) & \longrightarrow & \prod_{w|v} K_V[x]/(f_w) \\
\downarrow & & \downarrow \\
L \otimes_K K_V & \longrightarrow & \prod_{w|v} L_w
\end{array}
\]

Top is an iso by CRT.
Right: $x \to a_w$, an iso, because $K_V[x]/(f_w) \cong K_V(a_w) = L_w$.
Left: $x \to a \otimes l$, iso, because $K_0[x]/(f) \cong K(a) = L$ (extension of scalars!)

Everything commutes, so bottom is an iso also.
Given $L = K(\alpha)$ where $\alpha$ is a zero of $f(x) \in K[x]$. Write, in $K_v$, $f(x) = f_1(x)^{\nu_1} \cdots f_r(x)^{\nu_r}$ (where each $\nu_i$ is 1 in the separable case).

How to get a $K$-embedding $\tau : L \rightarrow \overline{K_v}$?

$\tau : L \rightarrow \overline{K_v}$

$\alpha \mapsto \beta$, where $\beta$ is a zero of $f(x)$ in $\overline{K_v}$.

Two embeddings $\tau, \tau'$ are conjugate iff the $\beta$'s chosen are roots of the same irreducible $f_i(x)$.

Theorem 8.2. With the above, the valuations $\nu_1, \ldots, \nu_r$ extending $v$ to $L$ are in bijection with the $f_i$ above.

Moreover, we see how to get them:

Take $\alpha_i \in \overline{K_v}$ a zero of some $f_i$.

$\tau_i : L \rightarrow \overline{K_v}$ a $K$-embedding.

$\alpha \mapsto \alpha_i$

Then $\nu_i = v \circ \tau_i$, and $\tau_i$ extends to an isomorphism $\overline{\tau_i} : L_{\nu_i} \rightarrow \overline{K_v(\alpha_i)}$.

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Valuations and tensor products.

With above, get a hom. $L \otimes K_v \rightarrow \prod_{i=1}^{r} L_{\nu_i}$

$a \otimes b \mapsto ab$.

What is $L \otimes K_v$? First of all, it's a $K_v$-vector space. But it's a $K_v$-algebra, multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

Do this for all $\nu_i$, obtain $L \otimes K_v \rightarrow \prod_{i=1}^{r} L_{\nu_i}$. 

Given \( \tau \) and \( \tau' \), with \( \tau' = \sigma \circ \tau \) for \( \sigma \in \text{Gal}(\overline{K_v}/K_v) \),
(two embeddings conjugate over \( K_v \))

Well, \( \overline{\nu} \) is the only extension of \( \nu \) from \( K_v \) to \( \overline{K_v} \)
So \( \overline{\nu} = \overline{\nu} \circ \sigma \). So \( \overline{\nu} \circ \tau = \overline{\nu} \circ (\sigma \circ \tau) \).

Conversely: Given \( \tau, \tau' : L \hookrightarrow \overline{K_v} \) \( K \)-embeddings, s.t.
\( \overline{\nu} \circ \tau = \overline{\nu} \circ \tau' \).

Define a \( K \)-isomorphism \( \sigma : \tau L \rightarrow \tau' L \)
\( \sigma = \tau' \circ \tau^{-1} \).

Then \( \sigma \) extends to a \( \overline{K_v} \)-isomorphism
\( \sigma : \tau L \cdot K_v \rightarrow \tau' L \cdot K_v \).

Why? \( \tau L \) is dense in \( \tau L \cdot K_v \), so every \( x \in \tau L \cdot K_v \) is
a limit \( x = \lim\limits_{n \to \infty} \tau \chi_n \) with \( \chi_n \in \tau L \).

With \( \tau' \chi_n = \sigma \tau \chi_n \), because of \( \overline{\nu} \circ \tau = \overline{\nu} \circ \tau' \) we have
\( \sigma \chi_n = \lim\limits_{n \to \infty} \tau' \chi_n \).

In other words, equality of valuations guarantees we
get a Cauchy sequence, Define \( \sigma \chi_n \) to be the RHS.

Easily checked, the map \( x \mapsto \sigma \chi_n \) does not depend on the
\( \chi_n \) chosen, so we get an isomorphism
\( \tau L \cdot K_v \stackrel{\sigma}{\longrightarrow} \tau' L \cdot K_v \) leaving \( K_v \) fixed.

This is our Galois element:

Extend \( \sigma \) (arbitrarily) to a \( \overline{K_v} \)-automorphism \( \overline{\sigma} \in \text{Gal}(\overline{K_v}/K_v) \).

Get \( \tau' = \overline{\sigma} \circ \tau \)
So \( \tau, \tau' \) one conjugate over \( K_v \).
1.6.1. Local and global fields.

Interested in the following:

\( L/K \) extension of global fields (here: number fields), \((L:K) \) finite \((\text{see Narkich for gen. case})\),

\( v: \) valuation on \( K \),

\( w: \) valuation on \( L \) extending \( v \).

How to get this?

Choose an embedding \( L \to K_v \).

Get a valuation \( v_\tau \) on \( K_v \).

Use this embedding to obtain \( w \) on \( L \).

Above extends to a continuous map \( Lw \to \overline{K_v} \).

\[
\begin{array}{c}
Lw = L K_v, \text{ and } (x|_w = \sqrt{N_{Lw/K_v}(x)}).
\end{array}
\]

**Theorem. (Extension Theorem 8.1)**

Given the above,

1. Every extension \( w \) of \( v \) arises as the composite \( w = v_\tau \) for some \( K \)-embedding \( \tau: L \to K_v \).

2. Two extensions \( v_\tau, v_{\tau'} \) are equal if and only if \( \tau \) and \( \tau' \) are conjugate over \( K_v \).

**Proof.** (1) Choose some \( w|_L \), form \( Lw \).

Choose some \( K_v \)-embedding \( \tau: Lw \to K_v \), then by construction \( v_\tau \) must coincide with \( w \).

Restricting to \( L \) gives what we want.