Important corollary.

**Approximation Theorem.** Let \( l_1, l_1, \ldots, l_n \) be pairwise inequivalent valuations. Given \( a_1, \ldots, a_n \in K \) and \( \epsilon > 0 \).

There exists \( x \in K \) s.t.

\[
| x - a_i |_i < \epsilon \quad \text{for all} \quad i = 1, \ldots, n.
\]

What does this mean?

Let \( K = \mathbb{Q} \), consider \( l_1, l_2, l_3, l_4, l_5, l_6 \), \( \epsilon = \frac{1}{10} \).

Let \( a_1 = 2, a_2 = 3, a_3 = 5 \).

Then there exists \( x \in \mathbb{Q} \),

\[
| x - 2 |_3 < \frac{1}{10}, \quad | x - 3 |_5 < \frac{1}{10}, \quad | x - 5 |_7 < \frac{1}{10}.
\]

If \( x \in \mathbb{Z} \), says same as \( x \equiv 2 \pmod{27} \),

\[
 x \equiv 3 \pmod{25}, \quad x \equiv 5 \pmod{49}.
\]

So it's like CRT.

But, maybe \( x \in \mathbb{Q} \).

Could also throw in the real valuation.

E.g., \( | x - \pi |_\infty < \frac{1}{10} \).

Here, certainly \( x \in \mathbb{Z} \) not good enough!

**Proof.**

**Claim.** There exists \( x \in K \) with

\[
| x |_i > 1, \quad | x |_j < 1 \quad \text{for} \quad j \neq 1.
\]

**Reference.**
Proof of claim for \( n = 2 \). (Two rotations)

Almost a tautology. By the extended prop.,

there are \( \xi, \beta \in K \) with

\[
|\xi| \leq 1 \quad |\xi|^2 \geq 1 \quad \text{(if } |\xi| \text{ we're done)}
\]

\[
|\beta| \leq 1 \quad |\beta|^2 \geq 1.
\]

and \( \left| \frac{\xi}{\beta} \right| < 1 \quad \left| \frac{\xi}{\beta} \right|^2 > 1 \).

Now, induct. Suppose

\[
|\beta_j| > 1 \quad |\beta_j| < 1 \quad \text{for } j = 2, \ldots, n-1.
\]

If \( |\beta_n| < 1 \)? done.

If \( |\beta_n| = 1 \)? Take \( \gamma' = \gamma^m \) where \( m \) is big,

\[
|\gamma| < 1 \quad |\gamma|^n > 1.
\]

If \( |\beta_n| > 1 \)? Look at \( \frac{\gamma^m}{1 + \gamma^m} \), converges to 1 w.r.t.

\( 1, 1_n \)

and \( 1, 1_n \)

converges to 0 w.r.t.

\( 1, 1_n \)

Choose \( \gamma' = \frac{\gamma^m}{1 + \gamma^m} \) for \( m \) big.

So the sequence \( \frac{\gamma^m}{1 + \gamma^m} \) converges to 1 in \( 1, 1_n \)

0 in \( 1, 1_n \)

(such very close).

Write \( w_1 \) for this, and similarly \( w_2, \ldots, w_n \).

Then, choose \( \gamma = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n \).

Then \( |x - a_1| = |a_1 (w_1 - 1) + a_2 w_2 + \cdots + a_n w_n| \)

is really small \( \text{and so } \epsilon \text{ for suitable } x \).
Prop. (3.7) Every valuation of $\mathbb{A}$ is equivalent to one of the valuations $1 \cdot |p|$ or $1 \cdot |10|$

Some general setup and results. (Same proofs as for $\mathbb{Z}_p$, $\mathbb{Q}_p$.) (See N., Ch. 3 - 4.)

Proposition. Let $K$ be a field with valuation $v(-)$ and absolute value $|\cdot| = q^{-v(-)}$ for some $q > 1$.

(Recall: different choices of $q$ : equiv. valuations)

The subset
$$O = \{ x \in K : v(x) \geq 0 \} = \{ x \in K : |x| \leq 1 \}$$

is a ring with group of units
$$O^* = \{ x \in K : v(x) = 0 \} = \{ x \in K : |x| = 1 \}$$

and unique maximal ideal
$$p = \{ x \in K : v(x) > 1 \} = \{ x \in K : |x| < 1 \}$$

The valuation is discrete if it has a smallest positive value $s$, and normalized if $s = 1$.

Dividing by $s$, can always pass to a normalized valuation.

The prime elements are those in $\mathfrak{p} / \mathfrak{p}^2$.

Writing $\pi$ for an arbitrary prime elt., every $x \in K^*$ can be written uniquely as $x = u \cdot \pi^m$ for $u \in O^*$, $m \in \mathbb{Z}$.

(If $v(x) = m$, then $v(x \pi^{-m}) = 0$ so it is a unit.)
2.5.

If \( v \) is a discrete valuation, then \( \mathcal{O} \) is a PID with a unique maximal ideal; i.e. a discrete valuation ring.

The ideals are \( p^n \), for \( n \in \mathbb{Z} \), and we have

\[
K^x = (\mathfrak{m}) \times \mathcal{O}^x.
\]

(In fact, \( K^x = (\mathfrak{m}) \times \bigwedge^{n-1} \mathfrak{m}^x \times \bigwedge^1 \mathcal{U}^x \),

where \( \mathfrak{m} \) are roots of unity, \( \mathfrak{m}^x \) is the principal units, \( \mathcal{U} \)

We let \( \hat{\mathcal{O}} \) be the completion of \( \mathcal{O} \) w.r.t. \( \mathfrak{m} \).

Then the maximal ideal of \( \hat{\mathcal{O}} \) is \( \hat{p} \), and

\[
\hat{\mathcal{O}} / \hat{p}^n \cong \mathcal{O} / p^n \quad \text{for every } n \geq 1.
\]

Moreover, we have an isomorphism and homeomorphism

\[
\mathcal{O} \longrightarrow \lim_{n} \mathcal{O} / p^n.
\]

So, the question is:

Given \( K / \mathcal{O} \), can cook up valuations on \( K \).

Complete with respect to them. Get “local fields”.

Can take the opposite approach. Start with \( \mathbb{Q}_p \).

Consider an algebraic extension. Do we get the same?

e.g. \( \sqrt{-1} \) of \( \mathbb{Q}_7 \).

\( \mathbb{Q}_7(i) \). Is it complete?

Indeed, is it the completion of \( \mathbb{Q}(i) \)? Yup!

\( \mathcal{O} \longrightarrow \mathbb{Q}_7 \)

Goal: understand extensions of valuations.
13.1. Extensions of local fields.

Def. A field $K$ is **local** if:
- it is complete w.r.t. a discrete valuation;
- it has a finite residue field.

**Theorem.** (N 2.5.2) These are precisely the finite extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p(C^1)$.

**Theorem.** They all satisfy Hensel's Lemma.
(See N 2.6, "Henselian fields").

**Applications of Hensel.**

**Prop.** We have $\mathbb{Q}_p \cong \mathbb{Z}_p^{1/p - 1}$ ($p - 1$ th roots of unity).

**Proof.** $(\mathbb{Z}_p^{\times})$ is an abelian group of size $p - 1$.
That means $a^{p - 1} \equiv 1 \mod p$ for all $a \in \mathbb{Z}_p^\times$.
So $x^{p - 1} - 1 \in \mathbb{Z}_p[x]$ splits completely in $\mathbb{F}_p[x]$.
By Hensel, it splits into distinct factors in $\mathbb{Z}_p[x]$ too.

**Prop.** Let $K$ be complete w.r.t. monoch. $1$. (e.g. $K = \mathbb{Q}_p$)
For every irreducible polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$ with $a_0 a_n \neq 0$, one has
$$|f| = \max\{ |a_0|, |a_n| \}.$$  
(Here $|f| = \max |a_i|$.)
In particular, writing $\mathcal{O}$ for the valuation ring of $K$, $a_n = 1$ and $a_0 \in \mathcal{O}$ imply $f \in \mathcal{O}[x]$. 
Proof. By multiplying through by an element of \( k \),
can assume \( f \in \mathcal{O}[x] \) and \( \|f\| = 1 \).

In the list \( a_0, a_1, a_2, \ldots \) let \( a_r \) be the first which has \( |a_r| = 1 \).

Then, mod \( \mathfrak{p} \) (\( \mathfrak{p} = \text{max ideal of } \mathcal{O} \)),

\[
f(x) \equiv a_r x^r + a_{r+1} x^{r+1} + \ldots + a_n x^n \mod \mathfrak{p}
\]

\[
\equiv x^r (a_r + a_{r+1} x + \ldots + a_n x^{n-r})
\]

If \( \max \{|a_0|, |a_n|\} < 1 \), this is a nontrivial factorization into coprime polynomials.

By Hensel it lifts from \( \mathcal{O}/\mathfrak{p} \) to \( \mathcal{O} \). **Contradiction**

**Big Theorem. (4.8)** Let \( K \) be complete w.r.t. \( \| \| \).

Let \( L/K \) be any algebraic extension. Then \( \| \| \) extends uniquely to \( L \) with

\[
\|a\| = \sqrt[n]{\text{N}_{L/K}(a)} \quad (n = [L:K] \text{ when } L/K \text{ finite})
\]

Then \( L \) is also complete.

Proof. Assume:
- \( \| \| \) is nonarchimedean (otherwise \( K \) is \( \mathbb{R} \) or \( \mathbb{C} \))
- \( L/K \) is finite. (Can assume WLOG: Prove for \( K(a) \),
take compositum over all \( a \in L \).)
3.3.

Notation: \( L = \langle \rangle \) (will prove \( \text{int} \) closure of \( \bigcirc \) in \( L \).
will prove: is valuation ring \( \text{of} \ L \).
\( K = \langle \rangle \) (valuation ring) \( \not\in \mathfrak{p} \) (unique max ideal).

Note. \( \bigcirc \) and \( \bigcirc \) are easy to confuse. Sorry.

Proof. (existence):

Let \( \bigcirc \) be \( \text{int} \) closure of \( \bigcirc \) in \( L \).

Claim: \( \bigcirc = \{ q \in L : N_{L/K}(q) \in \bigcirc \} \).

Proof of claim.

1. Given \( q \in \bigcirc \), satisfies a monic poly in \( \bigcirc \) norm is \( \pm (\text{its last coefficient})^m \) for some \( m \).

2. Given \( q \in L^* \) with \( N_{L/K}(q) \in \bigcirc \).

Let \( f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0 \in K[x] \)

\( \text{min poly of} \ q. \)

Then \( N_{L/K}(q) = \pm q_0^m \), so \( |a_0| \leq 1 \) (i.e. \( q_0 \in \bigcirc \)).

Use Proposition 4.7: \( f(x) \in \bigcirc \)[x].

By def., \( q \in \bigcirc \).

Now define \( |a| = \sqrt[n]{|N_{L/K}(a)|} \). (Note: if \( \beta \in K \), then \( \sqrt[n]{|N_{L/K}(\beta)|} = |\beta| \).

Easy: \( |a + 1| = 0 \iff q = 0 \)

\( |a \beta| = |a| |\beta| \).

Want to check strong triangle inequality

\( |a + \beta| \leq \max \{|a|, |\beta|\} \).

Restriction by \( \bigcirc \):

Assume \( \text{WLOG} \ 1 \not\in \bigcirc \), divide by \(|\beta| \),

enough to check

\( |a + 1| \leq \max \{|a|, 1\} \).

i.e. \( |a + 1| \leq 1 \) for \( \text{exact values} \) if \( |a| \leq 1 \).
By claim, this reduces to \( q \in O \Rightarrow q + 1 \in O \). But this is trivial. Integral elt are a ring.

Therefore, \( \{q \} = \mathfrak{m}((N_{L/K}(q)) \) defines a valuation on \( L \) which agrees with old valuation on \( K \).

Moreover, \( O \) is the valuation ring by our claim.

Uniqueness. Suppose \( l \cdot l' \) is another elt w/ valuation ring \( O' \).

Let \( \mathfrak{p}, \mathfrak{p}' \): max ideals of \( O, O' \).

Claim. \( O \subseteq O' \).

Proof. Note \( O, O' \) are both in \( L \) (by construction).

Given \( a \in O \setminus O' \) with min poly

\[ f(x) = x^d + a_d x^{d-1} + \ldots + a_0. \]

Then all the \( a \)'s are in \( O \), and \( a^{-1} \in \mathfrak{p}' \).

(because it is not in \( O' \))

Plug in \( a \).

\[ q = q^d + a_d^{-1} q^{d-1} + \ldots + a_0 q^{-1}, \]

\[ 0 = 1 + a_d^{-1} q^{-1} + \ldots + a_0 q^{-d}. \]

This is in \( \mathfrak{p}' \), so \( 1 \) is also, contradiction.

Thus, \( O \subseteq O' \), i.e. \( |q| \leq 1 \Rightarrow |q|' \leq 1 \).

By the approximation theorem, \( l \cdot l \) and \( l \cdot l' \) are equivalent, i.e. \( l \cdot l = (l \cdot l')^s \) for some \( s > 0 \).

Since they agree on \( K \), they are equal.
13.5. \( L \) is complete with respect to this valuation.

Proof omitted; see N. 2.4.9.

So, extend valuations from \( K \) to \( L \). \( [L : K] = n \).

For absolute values, \( |a| = \sqrt[n]{N_{L/K}(a)} \).

In terms of (additive valuations),

a valuation \( v \) on \( K \) extends to a valuation \( w \) on \( L \) satisfying

\[
w(a) = \frac{1}{n} v(N_{L/K}(a)).
\]

Note also, if \( v \) is normalized s.t. \( v(K^\times) = \mathbb{Q} \), then \( \frac{1}{n} \mathbb{Z} \leq w(L^\times) \leq \mathbb{Z} \).

Example. Let \( \mathfrak{p} = 7, K = \mathbb{Q}_p, L = \mathbb{Q}_p(\sqrt{p}) \).

Then for \( a \in L \), \( |a| = \sqrt[n]{N_{L/K}(a)} \).

In particular, \( |\sqrt{p}| = \sqrt[n]{N_{L/K}(\sqrt{p})} \)

\[
= \sqrt[n]{\sqrt{p} \cdot (-\sqrt{p})} = \sqrt{1-p} = \sqrt{p}.
\]

The same calculation gives \( w(\sqrt{p}) = \frac{1}{2} \),

where \( w \) is the extended valuation.

Example. Let \( p = 7, K = \mathbb{Q}_p, L = \mathbb{Q}_p(\sqrt{3}) \).

(Check: 3 is not a quad. residue.)

Then \( N_{L/K}(a + b\sqrt{3}) = a^2 - 3b^2 \).

Check: If this is divisible by 7, it is divisible by 7^2.

Thus \( w(L) = \mathbb{Z} \).
3.6. **Definition.**

The index \([w(L^x) : v(K^x)]\) is called the **ramification index** of \(L/K\).

Write \(e(wlv)\).

**Def.** Given \(L/K\) with valuation rings \(O \mid \Lambda\),
max ideals \(P \mid \mathfrak{p}\).

Have residue fields \(\Lambda : = \Omega/P\)
\(\kappa : = \alpha/\mathfrak{p}\).

As before \(K \hookrightarrow \Lambda\) and \(\Lambda\) is a finite ext.

The degree \([\Lambda : \kappa]\) is the **inertia degree** of \(L/K\).

Write it \(f(wlv)\).

**Theorem.** If \(\mathfrak{p}\) \(L/K\) is finite separable, \(v\) is a
disc. valuation on \(K\), \(w\) extends it, then

\([L : K] = e(wlv) \cdot f(wlv)\).

(Ponder: where did the \(g\) go?)
Last time.

Suppose $K$ is complete w.r.t. $\mathfrak{m}_L$.

$L/K$ alg. extension.

Then $L/K$ may be uniquely extended to $L_1$ with

$$|\alpha| = \sqrt{|N_{L/K}(\alpha)|}.$$

$L$ is again complete w.r.t. $\mathfrak{m}_L$.

In terms of additive valuations,

get a valuation $w$ prolonging the valuation $\nu$ on $K$,

with $w(\mathfrak{a}) = \frac{1}{n} \nu(N_{L/K}(\mathfrak{a}))$.

So,

$$\frac{1}{n} \nu(K^x) \cong w(L^x) \cong \nu(K^x).$$

**Def.** $e_w(\nu):= \left[\nu(L^x) : \nu(K^x)\right]$ is the **ramification index** of $L/K$ (of $w|\nu$).

Let $O$ and $\mathfrak{o}$ be the valuation rings,

$\lambda := O/\mathfrak{m}_L$ and $x := O/\mathfrak{m}_K$ the residue class fields.

We have an injection $K \hookrightarrow \lambda$:

$$O/\mathfrak{m}_K \rightarrow O/\mathfrak{m}_L$$

$$x \rightarrow x.$$  

Well defined because $\mathfrak{m}_K \cdot O \subseteq \mathfrak{m}_L$.

Injective because 1 is not in the kernel.

**Def.** $f(\nu) := [\lambda : x]$ is the **residue class degree**.
Let $\pi$ and $\pi'$ be prime elements of $O$ and $\mathfrak{p}$. Then $w(L^x) = w(\pi) \cdot \mathbb{Z}$, $w(\mathfrak{p}^x) = w(\pi') \cdot \mathbb{Z}$.

$$e = [w(\pi) : w(\pi')] \cdot \mathbb{Z},$$

so that $w(\pi') = e \cdot w(\pi)$, i.e.

$$\pi = e \cdot \pi^e \text{ for some } \varepsilon \in O^x.$$

In particular, we see that $\mathfrak{p}^0 = \pi^0 = \pi^e \mathfrak{p}^e \mathfrak{p}^e = \mathfrak{p}^e$,

i.e. $\mathfrak{p} = \mathfrak{p}^e$.

**Theorem.** Assume $L/K$ is finite separable and 1:1 discrete. Then $[L : K] = ef$.

**Proof.** (1) show $ef \leq [L : K]$.

Let $w_1, \ldots, w_f$ be a basis for $L/K$. (i.e. they live in $L^x$)

\[ w_1, \ldots, \pi^{e-1} \pi^e \ldots \pi^e \text{ represent } \pi \text{ representing all the cosets of } [w(L^x) : w(\mathfrak{p}^x)] .\]

Want to show the $w_j \cdot \pi^i$ are (1) linearly independent $/K$

(2) a basis of $L/K$.

To show (1), write

$$\sum_{i=0}^{e-1} \sum_{j=1}^f a_{ij} w_j \pi^i = 0 \quad a_{ij} \in \mathbb{K} .$$

If not all $a_{ij}$ are 0, then some $s_i := \sum_{j=1}^f a_{ij} w_j$ is not zero.

(because the $\pi^i$ are certainly linearly independent over $K$.)
4.3. Claim. If \( s_i \neq 0 \) then \( w(s_i) \in v(K^x) \).

**Proof.** Given \( \sum_{j \leq i} a_{ij} w_j = 0 \), divide by the \( a_{iv} \) of minimum value.

Get \( s_i' = \frac{s_i}{a_{iv}} = \sum_{j} a_{ij} \frac{w_j}{a_{iv}} \) These are in \( L \)

These are in \( \mathfrak{o} \subseteq K \).

The \( w_j \) represent a basis for \( \mathfrak{l} / \mathfrak{k} \).

Therefore, \( s_i' \) can only be \( 0 \) (mod \( \mathfrak{p} \)) if all \( a_{ij} \) are \( 0 \) (mod \( \mathfrak{p} \)).

But we divided by \( a_{iv} \) of min value, so \( a_{iv} = 1 \),

so \( s_i' \neq 0 \) (mod \( \mathfrak{p} \)) and so is a unit in \( \mathfrak{O} \).

This implies \( w(s_i) = w(a_{iv}) + v(K^x) \) (because \( a_{iv} \in K \)).

[Note: We're really using everything!]

Now, we had a sum \( O = \sum_{i=0}^{e-1} s_i \mathfrak{I}^i \).

Two summands must have the same valuation,
because \( w(x) \neq w(y) \Rightarrow w(x+y) = \min \{ w(x), w(y) \} \).

However, the \( s_i \) all have valuations in \( v(K^x) \)
the \( \mathfrak{I}^i \) all represent distinct cosets of

This is a contradiction.

Proves linear independence, i.e. \( ef \subseteq [L:K] \).
T4.9. (2). Need:

Nakayama's Lemma. Let A be a local ring with maximal ideal \( \mathfrak{m} \).

Let \( M \) be an \( A \)-module, \( N \subseteq M \) a submodule with \( M/N \) finitely generated.

Then, \( M = N + \mathfrak{m}M \rightarrow M = N \).
(Proof. Exercise)

To do (2), consider the \( A \)-module

\[
M := \sum_{i=0}^{e-1} \sum_{j=1}^{f} a_{w_j} T_i^j.
\]

Will argue that \( M = 0 \), i.e., \( \{w_j T_i^j\} \) are not only linearly dependent, but an integral basis for \( 0\mathfrak{m} \).

Write \( N = \sum_{j=1}^{f} a_{w_j} \),

\[
M = N + T_1 N + T_1^2 N + \ldots + T_1^{e-1} N.
\]

Then we have \( 0 = N + \mathfrak{m}N \).

Why? For \( \neq 0 \), look at \( a \) mod \( \mathfrak{m}T_1 \).

Get \( a_1 w_1 + \ldots + a_f w_f \) (mod \( \mathfrak{m}T_1 \)) for some \( a_i \neq 0 \).

Residue can be represented by sum of valuation \( \theta \), and all such elts. are spanned by a basis of \( \lambda : K \).
(In other words: \( w_1, \ldots, w_f \) are a basis for \( 0/P \) over \( \mathfrak{m}/P \).

(\text{So: } a_i \text{ one only determined up to } \mathfrak{m}.)

So, \( 0 = N + \mathfrak{m} = N + \mathfrak{m}(N + \mathfrak{m}O) = \ldots = N + T_1 N + \ldots + T_1^{e-1} N + T_1^e O \),

i.e., \( 0 = M + P^e = M + \mathfrak{m}O \).

Now \( 0 \mathfrak{m}O \) is finitely generated (has an integral basis), so Nakayama applies and \( 0 \subset M \).
Def. \( L/K \) (finite ext. of \( @p \)) is \underline{unramified} if

\[
[L : K] = [\lambda : k],
\]
i.e. \( e(L/K) = 1 \).

An arbitrary algebraic extension \( L/K \) is unramified if it is a union of finite unramified extensions.

Prop. (7.2) Given \( L|k, k'||k \) inside a fixed alg closure \( F \). Then,

\[ L|k \text{ unramified } \Rightarrow L \cdot k'||k \text{ unramified.} \]

Proof. Write \( L' = L \cdot k' \)

use the notation \( \theta, p, k, \theta', p', k', \theta, p, \lambda, \theta', p', \lambda' \).

Can argue just for finite extensions.

By the \underline{primitive element theorem} \( \lambda = k(\bar{\bar{a}}) \) for some \( a \in \theta \).

Write \( f(x) \in \theta[x] \text{ min. poly of } a \). \( \bar{f}(x) = \bar{f}(x) \text{ mod } p \in k(x) \).

Then

\[
[\lambda : k] \leq \deg(\bar{f}) = \deg(f) = [k(\bar{a}) : k] \leq [L : K] = [\lambda : k],
\]
so \( L = k(\bar{a}) \) and \( \bar{f} \) is the min. poly of \( \bar{a} \) over \( k \).

So \( L' = k'(\bar{a}) \).

Why is \( L'|k' \) unramified?

Let \( g(x) \in \theta' \cdot [x] \text{ min. poly of } a \text{ over } k' \).

\( \bar{g}(x) = g(x) \text{ mod } p' \in k'[x] \).

Note that \( \bar{g}(x) \) is a factor of \( \bar{f}(x) \).

By Hensel's Lemma \( \bar{g}(x) \) is irreducible .

(If it factored, would lift to a factorization of \( g(x) \).

So \( [\lambda' : k'] \leq [L' : k] = \deg(g) = \deg(\bar{g}) = [k'(\bar{a}) : k'] \leq [\lambda' : k'] \).

So \( [L' : k'] = [\lambda' : k'] \), \( L'|k' \) \underline{unramified}.
4.6. **Cor.**

If \( L' \mid K \) is an unramified extension and \( L \subseteq L' \), then \( L \mid K \) is also unramified.

**Proof.** By prop., \( L' \mid L \) is unramified.

Have \( [L' : K] = [\lambda_{L'} : \kappa] \)
\[ [L' : L] = [\lambda_{L'} : \lambda_L] \cdot \]

Since field degrees are multiplicative, \( L \mid K \) is ur. (i.e. \( [L : K] = [\lambda_L : \kappa] \).

**Cor.** If \( L \) and \( L' \) are unramified over \( \kappa \), so is \( LL' \).

**Proof.** \( LL' \mid L \) is unramified, with
\[ [\lambda_{L'} : \kappa] = [L' : K] \]
\[ [\lambda_{LL'} : \lambda_{L'}] = [LL' : L'] \cdot \]
(Use: separability is transitive)

**Def.** Fix an algebraic closure \( \overline{K} \) of \( K \).

Then the composite of all unramified subextensions \( L \subseteq \overline{K} \) of \( K \) is the maximal unramified extension \( \overline{L} \) of \( K \).

**Prop.** (7.5) The residue class field of \( T \) is \( \overline{\kappa} \) (\( = \overline{\mathbb{F}_p} \)).

Moreover, \( v(T^\times) = v(K^\times) \).

**Proof.** See Neukirch, but this is not hard.

(Tame ramification: 7.6, 7.7, 7.8, 7.9, 7.10, 7.11)