Problem Set 6 – Arithmetic Geometry, Frank Thorne (thorne@math.sc.edu)

Due Monday, March 28, 2016

(Choose one.)

(1) A special case of **Schanuel's Theorem** says that

$$C(\mathbb{P}^N(\mathbb{Q}), B) := \{P \in \mathbb{P}^N(\mathbb{Q}) \ : \ H(P) \leq B\} \sim \frac{2^N}{\zeta(N+1)} B^{N+1}.$$

Prove this. A step-by-step proof is outlined in the exercise set; if you want to build your analytic number theory chops, **don't read it.**

(2) Let $\phi : \mathbb{P}^N \to \mathbb{P}^M$ be a morphism of degree d, i.e., one defined by

$$\phi(P) = [f_0(P) : f_1(P) : \cdots : f_M(P)]$$

where all the f_i are homogeneous polynomials of degree d.

(a) Prove that for all $P \in \mathbb{P}^N$ we have the upper bound

$$h(\phi(P)) \ll dh(P).$$

The implied constant may depend on everything other than P. (You may see pp. 208-209 of Silverman but I encourage you to come up with your own proof!)

(b) Note that the f_i cannot have any common zeroes (otherwise ϕ is only a rational map). As such the set

$$\{Q \in \mathbb{A}^{N+1}(\mathbb{Q}) \ : \ f_0(Q) = \cdots = f_M(Q) = 0\}$$

consists of only the single point $(0, \ldots, 0)$. Use Hilbert's Nullstellensatz to argue that the ideal generated by $f_0, \cdots, f_M$ contains some power of each of the X_i.

(c) Prove the lower bound

$$h(\phi(P)) \gg dh(P).$$

This is more difficult, so you might choose to follow Silverman closely. (This is all in VIII.5, in the first edition more specifically pp. 209-210.) If you do, assume that you are working over $\mathbb{Q}$ and simplify Silverman's proof by omitting any details irrelevant to this special case.)