Problem Set 2 – Arithmetic Geometry, Frank Thorne (thorne@math.sc.edu)

Due Friday, January 29, 2016

Choose one. (Not that I discourage you from solving all three.)

(1) The *Veronese map* $v : \mathbb{P}^2 \to \mathbb{P}^5$ is defined by

$$[a : b : c] \longrightarrow [a^2 : ab : b^2 : ac : bc : c^2].$$

(a) Prove that the image of v (the *Veronese surface*) is a subvariety of $\mathbb{P}^5$ and that v defines an isomorphism onto its image.

(b) Identifying $\mathbb{P}^2$ and $\mathbb{P}^5$ with the spaces of lines and conics respectively, prove that the Veronese surface consists of those conics which are *degenerate*, i.e., which have a repeated factor. Conclude that the set of degenerate conics forms a subvariety of $\mathbb{P}^5$.

(c) Prove the same conclusion in a different way: First, demonstrate that there is a bijection between conic sections and 3×3 symmetric matrices, where the conic is given by

$$\begin{bmatrix} x & y & z \end{bmatrix} M \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Show that the degenerate conics consist of all symmetric matrices of rank 1, and that these form a subvariety of the space of all such matrices.

(d) Using the same identification with matrices, prove that *reducible* conic sections correspond to symmetric matrices of rank 2, and therefore themselves form a variety.

(2) (a) Prove the projective case of the *fundamental theorem of algebra*: given a homogeneous polynomial in 2 variables of degree d, then it factors (over $\mathbb{C}$) as a product of linear factors. Moreover, the roots of these linear factors are uniquely defined in $\mathbb{P}^1$.

(Recall that $[\alpha : \beta]$ is the unique root in $\mathbb{P}^1$ of the linear polynomial $\beta x - \alpha y$.)

You may of course rely on the usual fundamental theorem of algebra in your proof. Do not assume anything about your polynomial, except that it has at least one nonzero coefficient.

(b) If L is a line and $V(f)$ is a projective curve of degree d, define the intersection multiplicity of L with $V(f)$ at a point $P = [x_0 : y_0 : z_0]$ as follows: at least one of the coefficients defining L is nonzero, so we can solve for one of the variables and eliminate it from both P and f. Then the intersection multiplicity is the multiplicity of P (suitably interpreted) as a root of the reduced f.

Make this precise and prove that it does not depend on any of the choices made. Further, prove that the interseciton multiplicity remains the same if both L and $V(f)$ are changed by the same projective change of coordinates.

(3) (Presentation taken from Garrity et al., *Algebraic Geometry: A Problem Solving Approach.*)

(a) Consider eight distinct points P_1 through P_8 in $\mathbb{P}^2$, so that no four are collinear and no seven are on any conic. Let F be a generic cubic polynomial with unknown coefficients a_1 through a_{10}. The system of simultaneous equations

$$F(P_1) = F(P_2) = \cdots = F(P_8) = 0$$

is a system of eight linear equations in these ten unknowns. Prove that the vector space of solutions to these equations has dimension 2 by considering each of the following cases.

- (i) The eight points are in *general position*, which means that no three are collinear and no six are on a conic.
- (ii) Three of the points are collinear.
- (iii) Six of the points are on a conic.

(b) Show that there are two linearly independent cubics F_1 and F_2, so that any of cubic passing through the eight points P_1 through P_8 has the form $\lambda F_1 + \mu F_2$. Conclude that for any collection of eight points with no four collinear and no seven on a conic, there is a *unique* ninth point P_9 such that *every* cubic curve passing through the eight given points must also pass through P_9.

(c) Conclude the statement of the Cayley-Bacharach theorem given in class and in the lecture notes.