A special case of Schanuel’s Theorem says that
\[
C(\mathbb{P}^N(\mathbb{Q}), B) := \{ P \in \mathbb{P}^N(\mathbb{Q}) : H(P) \leq B \} \sim \frac{2^N}{\zeta(N + 1)} B^{N+1}.
\]

The following exercise outlines a proof of this theorem.

(a) Prove that the set to be counted is in an exactly 2-to-1 correspondence with \(N + 1\)-tuples of integers \((x_0, x_1, \ldots, x_N)\), where \(x_i \in [-B, B]\) for each \(i\), and the \(x_i\) do not all share a common factor.

(b) Write \(C(N, B)\) for the number of \(N + 1\)-tuples of integers \((x_0, x_1, \ldots, x_N)\), where \(x_i \in [-B, B]\) for each \(i\) (but with no ‘no common factor’ condition). Prove that
\[
C(N, B) = (2B)^{N+1} + O(B^N).
\]

(Be sure to justify that the constant implied by the \(O\)-notation is in fact independent of \(B\). It may depend on \(N\) however. You might wish to be especially careful and actually compute the constant implied in the error term.)

(c) Let \(\mu(d)\) be the Möbius function, equal to \((-1)^{\omega(d)}\) if \(d\) is squarefree, where \(\omega(d)\) denotes the number of prime factors of \(d\), and equal to zero otherwise.

Prove that, for any positive integer \(n\), \(\sum_{d|n} \mu(d)\) is equal to 1 if \(n = 1\) and zero otherwise.
(Hint: the sum can be rewritten as \(\prod_{p|n} (1 + \mu(p))\), where the product is over all primes dividing \(n\). Why is this?)

(d) Write \(C(N, B, d)\) for the number of \(N + 1\)-tuples of integers \((x_0, x_1, \ldots, x_N)\), where \(x_i \in [-B, B]\) for each \(i\), such that \(d\) divides all the \(x_i\). Explain why \(C(N, B, d) = C(N, B/d)\) and deduce an estimate for \(C(N, B, d)\).

(e) Prove that
\[
C(\mathbb{P}^N(\mathbb{Q}), B) = \sum_{d=1}^{B} \mu(d)C(N, B, d)
\]
and use your previous estimates to conclude that
\[
C(\mathbb{P}^N(\mathbb{Q}), B) = 2^N B^{N+1} \sum_{d=1}^{B} \mu(d) \frac{d}{d^{B+1}} + o(B^{N+1}).
\]

(f) Prove that
\[
\sum_{d=B+1}^{\infty} \frac{\mu(d)}{d^{N+1}} = o(1)
\]
and that
\[
\sum_{d=1}^{\infty} \frac{\mu(d)}{d^{N+1}} = \frac{1}{\zeta(B + 1)}.
\]
Conclude that

\[ \sum_{d=1}^{B} \frac{\mu(d)}{d^{N+1}} = \frac{1}{\zeta(N + 1)} + o(1). \]

(g) Conclude the statement of Schanuel’s theorem.

Note that Schanuel proved his result where \( \mathbb{Q} \) is replaced with any number field, where the analysis became more difficult. For definitions of height functions in number fields, see Chapter 8.5 of Silverman.