4.1.

Last time: Restriction ~~of~~ of scalars from $\mathbb{C}$ to $\mathbb{R}$.

Proposition. There is an injection $GL_n(\mathbb{C}) \xrightarrow{Res_{\mathbb{C}/\mathbb{R}}} GL_{2n}(\mathbb{R})$
$M_n(\mathbb{C}) \longrightarrow M_{2n}(\mathbb{R})$
and its image consists of matrices commuting with

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

(The exact matrix depends on the particular ~~injection~~ isomorphism $\mathbb{C}^n \cong \mathbb{R}^{2n}$.
Here $(x_1+iy_1, \dots, x_n+iy_n)$
$\to (x_1, \dots, x_n, y_1, \dots, y_n)$

There is also an injection $M_n(\mathbb{H}) \longrightarrow M_{2n}(\mathbb{C})$
(and thus also $\longrightarrow M_{4n}(\mathbb{R})$.)

Represent a quaternion $a + b\vec{i} + c\vec{j} + d\vec{k}$ as

$$\cancel{\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}}$$

ARGH!! why don't books agree on their conventions??

$$\begin{pmatrix} a+di & -b-ci \\ b-ci & a-di \end{pmatrix}$$

and expand everything out in blocks.

Exercise. Characterize the image.

So the nice definition is: The set

$$\{A \in GL_n(K) : \langle Ax, Ay\rangle = \langle x, y\rangle \text{ for all } x, y\} \in K^n$$

is the orthogonal group $O(n) = O_n(\mathbb{R})$ when $K = \mathbb{R}$,
the unitary group $U(n) = U_n(\mathbb{C})$ when $K = \mathbb{C}$,
and the compact symplectic group $Sp(n)$ otherwise.

4.2. Topological properties of Lie groups.

We give $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$ and $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ the Euclidean (metric topology)

and any subgroups the subspace topolcgy.

Heine - Borel Theorem $\Rightarrow$ Any matrix Lie group $G \leq M_n(\mathbb{C})$ is ~~cl~~ compact if it is closed and bounded.

Proposition. $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $\underbrace{Sp(n)}_{\text{complex form}}$

are all compact.

Proof. They are all closed; for example

$$O(n) = \bigcap_{x,y \in \mathbb{R}^n} \{ A : \langle Ax, Ay \rangle = \langle x, y \rangle \}$$

$$= \bigcap_{x,y \in \mathbb{R}^n} \{ A : \cancel{\langle Ax, Ay} \langle (A-I)x, (A-I)y \rangle = 0 .$$

Inverse image ~~under~~ of 0 under a polynomial map.

They are also all bounded: they all satisfy $\bar{A}^T A = I$ and so have orthonormal columns and hence all entries bounded by 1.

By contrast, $SL(n)$ and $GL(n)$ are not compact.

4.3. Def. A matrix Lie group is path connected (book: connected) if for all $A, B \in G$ there is a continuous path $A(t)$, $a \le t \le b$ lying in G with $A(a) = A$, $A(b) = B$.

(Note: we could demand $a=0$, $b=1$ if we liked)

Same as demanding a function $f : [a,b] \to G$ as above.

(Later: path connected ~~for~~ $\iff$ connected for Lie groups)

Def. If G is a matrix Lie group (really, any manifold), the ~~connected~~ component of $x \in G$ is

$$\{ y \in G : \exists \text{ a cts path between } x \text{ and } y \}.$$

This relation partitions G into equivalence classes

Def. ~~W~~ The identity component is the component of $I \in G$.

Prop. If G is a matrix Lie group, its identity component G_0 is

(1) a subgroup of G

(2) normal.

Proof. (1) ~~Multiplication:~~ If $A, B \in G_0$, there are paths f, g connecting I to A, B respectively.

Then $f \cdot g$ connects I to AB

and $\frac{1}{f}$ connects I to A^{-1}.

(2) If $A \in G_0$, $B \in G$, $\exists$ a cts path f connecting I to A.

Then $t \to B \cdot f(t) \cdot B^{-1}$ connects I to BAB^{-1}.

4.4. Note: If $f, g : [a,b] \to GL_n(\mathbb{C})$ are continuous, so are $f \cdot g$ and $\frac{1}{f}$.

(cofactor expansion for inverse)
~~dets~~

Proposition. $GL_n(\mathbb{C})$ is connected for all $n \geq 1$.

Proof. Use Jordan form. Given $A \in M_n(\mathbb{C})$,

$A = CBC^{-1}$ for some C with

$$B = \begin{pmatrix} \begin{matrix} \lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_1 \end{matrix} & & \\ & \begin{matrix} \lambda_2 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_2 \end{matrix} & \\ & & \text{etc.} \end{pmatrix}$$

We just care that it's upper triangular.

Now B is in the same conn cpt. as 1.

~~Define~~ a path ~~$[0,1] \to GL_n(\mathbb{C})$~~

Choose your favorite continuous fns. $\lambda_i : [0,1] \to \mathbb{C}^\times$

$\lambda_i(0) = 1$

$\lambda_i(1) = \lambda_i$

and Define
$$B(t) = \begin{pmatrix} \begin{matrix} \lambda_1(t) & t & & \\ & \ddots & \ddots & \\ & & & t \\ & & & \lambda_1(t) \end{matrix} & \\ & \text{etc.} \end{pmatrix}$$

Get a path connecting I and B.

So $B \in G_0$.

By normality, so is CBC^{-1}.

4.5 . Prop. $SL_n(\mathbb{C})$ is conn'd for all $n \geq 1$.

Same; choose the λ_i with $\prod_{i=1}^{k} \lambda_i(t) \equiv 1$.

Prop. $U(n)$ and $SU(n)$ are connected.

Proof. Recall that unitary matrices satisfy $A^* A = I$

and so are normal (def: A commutes w/ adjoint)

Spectral Theorem. If M is a normal matrix, then M has an orthonormal basis of eigenvectors. (if and only if!)

Proof. Change of basis $\Rightarrow$ make it upper triangular. Now apply the definitions.

So: Any unitary matrix is diagonalizable and we win as before.

Def. If G, H are matrix Lie groups then $\Phi : G \to H$ is a Lie group homomorphism if it is a homomorphism and continuous.

(Same with isomorphisms.)

Examples. $GL(n, \mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times$

$\mathbb{R} \longrightarrow SO(2)$

$\theta \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Wait. What?? Is $\mathbb{R}$ a Lie group?
$GL_1^+(\mathbb{R}) := \{[x] \in GL_1(\mathbb{R}) : x > 0\}$ is, and $\mathbb{R} \underset{\sim}{\xrightarrow{\exp}} GL_1^+(\mathbb{R})$.

4.6. A more interesting example.

Prop. There is a Lie group hom $SU(2) \longrightarrow SO(3)$ which is 2-1 and onto.

Should you believe it?

$$SU(2) = \left\{ Q = \begin{pmatrix} a+di & -b-ci \\ b-ci & a-di \end{pmatrix} : \underbrace{\det Q = 1}_{a^2+b^2+c^2+d^2 = 1} \right\}$$

= unit quatenions.

How do you get a map to $SO(3)$??

Let $V = \{ X \in M_2(\mathbb{C}) : \underbrace{X^* = X}_{\text{("self-adjoint")}}, \ \mathrm{tr}\, X = 0 \}$.

Then any X looks like $\begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_1 \end{pmatrix}$

and V is the set of all such. So can identify $V \cong \mathbb{R}^3$.

Claim. The inner product is given by the trace form:

$$\langle X_1, X_2 \rangle = \frac{1}{2} \mathrm{trace}\,(X_1 X_2)$$

i.e. $\frac{1}{2} \mathrm{trace} \left(\begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix} \begin{pmatrix} x_1' & x_2' + ix_3' \\ x_2' - ix_3' & -x_1' \end{pmatrix} \right)$

$$= x_1 x_1' + x_2 x_2' + x_3 x_3'.$$

So: For each $U \in SU(2)$, define $\Phi_U : V \to V$

$$X \to U X U^{-1}.$$

Why is the image in V? (U is unitary!) (X is self-adjoint)

$$(UXU^{-1})^* = (U^{-1})^* X^* U^* = U X^* U^{-1} = U X U^{-1}$$

4.7 We have that Φ_U is clearly a homomorphism, and

$$\frac{1}{2}\operatorname{trace}\left((UX_1U^{-1})(UX_2U^{-1})\right) = \frac{1}{2}\operatorname{trace}(UX_1X_2U^{-1})$$
$$= \frac{1}{2}\operatorname{trace}(X_1X_2).$$

So preserves the inner product.

Get a homomorphism $SU(2) \to O(3)$.

Why into $SO(3)$? Because $SU(2)$ is connected!

Clearly $-I$ is in the kernel. Can check: That's all.

Finally, why is it onto? Could get this out. But...

Example. Suppose $U = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$.

$$U \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix} U^{-1} = \begin{pmatrix} x_1' & x_2' + ix_3' \\ x_2' - ix_3' & -x_1' \end{pmatrix}$$

where: $x_1' = x_1$

$$x_2' + ix_3' = e^{i\theta}(x_2 + ix_3)$$
$$= (x_2\cos\theta - x_3\sin\theta) + i(x_2\sin\theta + x_3\cos\theta).$$

$U \longrightarrow$ rotation by θ in (x_2, x_3)-plane.

4.8.

Some facts (exercises to prove).

1. Every element $A \in SO(3)$ is rotation about some axis.

Axis determined by an <u>eigenvector</u> w/ EV 1.

2. With V our $\mathbb{R}^3$, can write

$$X = U_0 \begin{pmatrix} x_1 & 0 \\ 0 & -x_1 \end{pmatrix} U_0^{-1} \quad \text{for some } U_0 \in U(2).$$

Plane orthogonal to this is space of matrices

$$X' = U_0 \begin{pmatrix} 0 & x_2 + i x_3 \\ x_2 - i x_3 & 0 \end{pmatrix} U_0^{-1} \qquad (*)$$

and we have $\quad U = U_0 \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} U_0^{-1}$

with $UXU^{-1} = X$,

$U\{\text{space of } X'\}U^{-1} = \{\text{space of } X'\}$

$UX'U^{-1}$ = same as $(*)$, but with (x_2, x_3) rotated by θ.

<u>So</u>: U maps to rotation by θ in the plane perpendicular to V_1, so <u>U maps to our A.</u>

5.1. Last time: the topology of Lie groups.

Prop. If G is a matrix Lie group, its identity component G_0 is a normal subgroup of G.

Proposition. $GL_n(\mathbb{C})$ is path connected.

Proof. Given $A \in M_n(\mathbb{C})$, $A \sim B$ w/ B upper triangular by Jordan form.

Compute an explicit path from I to B.

Same for $U(n)$ and $SU(n)$.

Def. If G, H matrix Lie groups then $\Phi: G \to H$ is a Lie group homomorphism if it is a homomorphism and continuous.

Examples. 1. $GL(n, \mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times$

2. $GL_1^+(\mathbb{R}) \overset{\exp}{\underset{\sim}{\leftarrow}} \mathbb{R} \longrightarrow SO(2)$

$$\theta \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

3. (symmetric square)

$$GL(2) \longrightarrow GL(3).$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} av+bw \\ cv+dw \end{pmatrix}.$$

Apply the map $v \to av+bw$, $w \to cv+dw$ to the vector space $\mathrm{Span}\{v^2, vw, w^2\}$.

5.2.

Proposition. There is a Lie group homomorphism

$$SU(2) \longrightarrow SO(3)$$

which is surjective and exactly 2-1.

Recall. $SU(2) = \{ A \in GL_2(\mathbb{C}) : A^* = A^{-1}, \det A = 1 \}$

$$= \left\{ \theta = \begin{pmatrix} a+di & -b-ci \\ b-ci & a-di \end{pmatrix} : \det \theta = 1 \right\}.$$

Let $\mathbb{H}^1$ be the unit quaternions

$\{ a + bi + cj + dk \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1 \}$

Proposition. These groups are isomorphic (the isomorphism given by letters having the same meaning).

You can regard either of these as being the unit 3-sphere S^3.
So S^3 is a group.
(If you think this is obvious: try to find a group structure on S^2.)

A pure imaginary quaternion is one with $a = 0$
or equivalently $\bar{x} = -x$.

Check. If $x \in \mathbb{H}$ is pure imaginary, and $|x| = 1$ then $x^2 = -1$.

So: if $t \in \mathbb{H}$, can write

$$t = \cos\theta + \underbrace{u}_{\text{pure imaginary quadratic with norm 1}} \sin\theta$$

5.3.

Proposition. Let $t = \cos\theta + u\sin\theta$ unit quotenion
$x = a\vec{i} + b\vec{j} + c\vec{k}$ pure imag. quotenion

Then $t^{-1}xt$ is a pure imaginary quotenion with
$|t^{-1}xt| = |x|$.
(Latter follows by multiplicativity of the norm.)

Proof. You can check it by hand.
Alternatively, the map $x \longrightarrow t^{-1}xt$ sends $\mathbb{R} \subseteq \mathbb{H}$ to $\mathbb{R}$.

Also the map is an isometry: for any $z \in \mathbb{H}$, any unit $t \in \mathbb{H}'$,
$|tz| = |z|$
so $|t^{-1}xt| = |x|$
and right or left multiplication by t preserves inner products: $\langle t^{-1}z_1, t^{-1}z_2\rangle = \langle z_1, z_2\rangle$
$\langle t^{-1}z_1 t, t^{-1}z_2 t\rangle = \langle z_1, z_2\rangle$

hence: $t^{-1}z_1 t \perp t^{-1}z_2 t \Longleftrightarrow z_1 \perp z_2$.

Thus the map sends $\mathbb{R}^\perp \longrightarrow \mathbb{R}^\perp$
i.e. pure imag. quotenions to themselves.

We thus get a representation

$$\begin{array}{ccc} SU(2) & \longrightarrow & GL(3) \\ \text{\rotatebox{90}{$\cong$}} & & \uparrow \mathbb{R} \\ \mathbb{H}' & & \\ t & \longrightarrow & \{x \to t^{-1}xt\}. \end{array}$$

And, we claimed $\langle t^{-1} z_1 t, t^{-1} z_2 t \rangle = \langle z_1, z_2 \rangle$

So this says exactly that the element $\{x \to t^{-1} x t\} \in GL(3)$ is in fact in $SO(3)$.

We can do better.

Proposition. If $t = \cos\theta + u \sin\theta$ $(u \in \mathbb{R}\vec{i} + \mathbb{R}\vec{j} + \mathbb{R}\vec{k})$ then the map $x \to t^{-1} x t$ is rotation through angle 2θ about axis u.

Proof. Claim 1. This map fixes multiples of u.

$$(\cos\theta - u\sin\theta)\, u\, (\cos\theta + u\sin\theta)$$

$$= u\cos^2\theta - u^2 \sin\theta\cos\theta + u^2\cos\theta\sin\theta - \underbrace{u^3}_{u^2=-1}\sin^2\theta = u.$$

So: look what conjugation does to a vector orthogonal to u.

Let $v \in \mathbb{R}\vec{i} + \mathbb{R}\vec{j} + \mathbb{R}\vec{k}$ be such a vector.

Then $u \cdot v$ $(= \langle u, v\rangle) = 0$.

Let $w = u \times v$ (usual vector calculus cross product).

Have $uv = -u \cdot v + u \times v$ (in general; check it)

$= u \times v$ here.

So $\{u, v, w\}$ is an ONB, $uv = w$, $vw = u$, $wu = v$
$vu = -w$, $wv = -u$, $uw = -v$.

5.5.

Remains to show $t^{-1}vt = v\cos(2\theta) - w\sin(2\theta)$
$t^{-1}wt = v\sin(2\theta) + w\cos(2\theta)$.

will prove the proposition.

$$t^{-1}vt = (\cos\theta - u\sin\theta)\, v\, (\cos\theta + u\sin\theta)$$

$$= v\cos^2\theta - uv\sin\theta\cos\theta + vu\cos\theta\sin\theta - uvu\sin^2\theta$$

use : $uv = -vu$, $u^2 = -1$

$$= v(\cos^2\theta - \sin^2\theta) - uv(2\sin\theta\cos\theta).$$

Done!

Similarly for w.

Now: Every element of $SO(3)$ is rotation about some axis (i.e. has 1 as an eigenvalue, prove it!)

The rotation is determined by the axis and angle except that $(u,\theta) \sim (-u,-\theta)$.

This proves the proposition!