15.8

Proof of Schur (Group case).

(1) Let $v \in \ker(\phi)$. Then

$$\phi(\Pi(A)v) = \Sigma(A)\phi(v) = 0.$$

So $\ker \phi$ is invariant. DONE

Well, almost: it's zero or one-to-one

and in the latter case $\mathrm{Im}(\phi)$ is invariant.

For all $w = \phi(v)$,

$$\Sigma(A)w = \Sigma(A)\phi(v) = \phi(\Pi(A)v).$$

(2) Given $\phi : V \to V$ with $\phi\Pi(A) = \Pi(A)\phi$.

Now ϕ has an eigenvalue $\lambda \in \mathbb{C}$ w/ eigenspace U.

So U is an invariant subspace, hence $U = V$.

(3) $\phi_1 \circ \phi_2^{-1}$ intertwining map $V \to V$. Use (2).

17.1

All rep'ns of $sl(2, \mathbb{C})$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

So the matrix of $\mathrm{ad}(H)$ is $\begin{bmatrix} 0 & & \\ & 2 & \\ & & -2 \end{bmatrix}$.

We already had the representations of binary n-ic forms discussed before.

Departing from Hall, write

$\mathrm{Sym}^n(\mathbb{C}^2) = \{$binary cubic forms of deg $n\}$.

15.9 = 17.2

Then, for each $n \geq 1$, $\mathrm{Sym}^n(\mathbb{C}^2)$ is an irrep of dim $n+1$.

Theorem. Every irrep of $\mathfrak{sl}(2, \mathbb{C})$ is one of these.

Proof. Given an irrep $(\pi, V) \ldots$

Lemma. Let u be an eigenvector of $\pi(H)$ with EV $q \in \mathbb{C}$.

Then

$$\pi(H)\pi(X)u = (q+2)\pi(X)u.$$

Proof.

$$\begin{aligned}\pi(H)\pi(X)u &= \pi(X)\pi(H)u + [\pi(H), \pi(X)]u \\ &= \pi(X)\pi(H)u + \pi([H, X])u \\ &\cancel{= \pi(X)\pi(H)qu} \\ &= \pi(X) \cdot qu + 2\pi(X)u.\end{aligned}$$

So: $\pi(X)$ sends eigenvectors to eigenvectors, and raises the EV by 2.

Similarly,

$$\pi(H)\pi(Y)u = (q-2)\pi(Y)u.$$

Proof of theorem. Given an irrep (π, V) of $\mathfrak{sl}(2, \mathbb{C})$.

Let u be an eigenvector for $\pi(H)$ (it must have one!) with eigenvalue q

Then by lemma, $\pi(H)\pi(X)^k u = (q+2k)\pi(X)^k u$.

We can't have infinitely many eigenvalues!

So, for some $N \geq 0$,

$$u_0 := \pi(X)^N u \neq 0, \quad \pi(X)^{N+1}u = 0.$$

Write $\lambda = q + 2N$,

$$\pi(H)u_0 = \lambda u_0 \qquad \pi(X)u_0 = 0.$$

15.10 = 16.6 = 17.3

Write now, for each k, $u_k = \pi(Y)^k u_0$, with $\pi(H)u_k = (\lambda - 2k)u_k$

Can check : $\pi(X)u_k = k\big(\lambda - (k-1)\big)u_{k-1}$ for all $k \geq 1$.

Let u_m be the last nonzero one. (Same argument as before)

Now $0 = u_{m+1} = \pi(X)u_{m+1} = (m+1)(\lambda - m)u_m$ so $\lambda = m$.

We have thus listed basis vectors for an invariant subspace of V (by irreducibility, is V itself)

Have to be linearly independent since they are EV's of $\pi(H)$ with distinct eigenvalues.

But we've just written down the entire representation.

Conversely, check that we really do have a representation.
Use our earlier construction, or define $\pi(H)$, $\pi(X)$, $\pi(Y)$ by the relations above and check the commutators.

16.7 = 17.4 A little plethysm. (See Fulton-Harris, Rep Thy, Ch 11)

Apparently this is a word. Analyze decompositions of rep'ns.

Example. Let $V \cong \mathbb{C}^2$ be the standard repn. of $sl(2)$.

i.e. $\pi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\begin{pmatrix} r \\ s \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$.

We have been discussing the symmetric power rep'ns $Sym^n(V)$.

e.g. $Sym^2(V) = \text{Span}\{x^2, xy, y^2\}$.

$V \otimes V = \text{Span}\{x\otimes x, x \otimes y, y \otimes x, y^2\}$

and $Sym^2(V) := V\otimes V / \langle v \otimes w - w \otimes v \rangle$.

We get a rep'n of G on $V \otimes V$:

~~$\pi(g)(v \otimes w) = (\pi(g)v) \otimes (\pi(g)w)$.~~ GAAAH!!! TOTALLY WRONG!

correct: $\pi(g)(v\otimes w) = \pi(g)v \otimes w + v \otimes \pi(g)\cdot w$.

This factors through the quotient above.

Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$, $x \to ax + cy$

$y \to bx + dy$

and so $\{x^2, xy, y^2\}$

$\downarrow$

~~$\{(ax+cy)^2, (bx+dy)(ax+cy), (bx+dy)^2\}$.~~

16.8 = 17.5

$$x^2 \to (ax+cy)\cdot x + x\cdot(ax+cy)$$
$$= 2x(ax+cy)$$

$$xy \to (ax+cy)\cdot y + \cancel{(bx+dy)} \; x\cdot(bx+dy)$$

This is all a bit weird.

So ask. what does $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ do?

Or, more precisely, $\pi(H)$

$$H(x\cdot x) = x\cdot H(x) + H(x)\cdot x = 2x^2.$$
$$H(x\cdot y) = x\cdot H(y) + H(x)\cdot y = xy - xy = 0$$
$$H(y\cdot y) = y\cdot H(y) + H(y)\cdot y = -2y^2.$$

Similarly, $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ sends $y \to x$ and kills x.

$$X(x\cdot x) = x\cdot X(x) + X(x)\cdot x = 0$$
$$X(x\cdot y) = x\cdot X(y) + X(x)\cdot y = x^2$$
$$X(y^2) = y\cdot X(y) + X(y)\cdot y = 2xy.$$

Similarly with $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

natural basis

So the point is these are all eigenvectors

$$\{x^2, \; xy, \; y^2\}$$

EV $\quad 2 \quad 0 \quad -2 \qquad$ (x-exponent) − (y-exponent)

$$\mathrm{Sym}^3 V = \mathrm{Span}\{x^3, x^2y, xy^2, y^3\}$$
$$\quad 3 \qquad 1 \qquad -1 \qquad -3$$

16.9 = 17.6

$$\mathrm{Sym}^4 V = \mathrm{Span}\{ \underset{4}{x^4},\ \underset{2}{x^3 y},\ \underset{0}{x^2 y^2},\ \underset{-2}{x y^3},\ \underset{4}{y^4} \}$$

What about $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$?

If the 3 basis vectors of $\mathrm{Sym}^2 V$ are v_1, v_2, v_3 then this is spanned by $\left\{ \begin{matrix} v_1 v_2, & v_1 v_3, & \text{and } v_2 v_3 \\ v_1^2, & v_2^2, & v_3 \end{matrix} \right\}$.

<u>So</u>:

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V) = \mathrm{Span}\{ x^2 \cdot x^2,\ x^2 \cdot xy,\ x^2 \cdot y^2,\ xy \cdot xy,\ xy \cdot y^2,\ y^3 \}$$

It is 6-dimensional, and we get a natural surjection

$$\begin{aligned} \mathrm{Sym}^2(\mathrm{Sym}^2 V) &\longrightarrow \mathrm{Sym}^4 V \\ q_1 \cdot q_2 &\longrightarrow q_1 q_2 \end{aligned}$$

whose kernel is spanned by $x^2 \cdot y^2 - xy \cdot xy$.

(Does your head hurt yet?)

Can we figure out the eigenvalues directly?

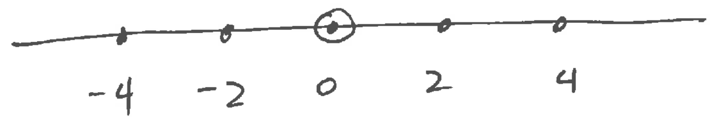
If v_1 and $v_2 \in \mathrm{Sym}^2 V$ are EV's of H with EV λ_1, λ_2,

$$\begin{aligned} H(v_1 \cdot v_2) &= v_1 \cdot H(v_2) + H(v_1) \cdot v_2 \\ &= \lambda_2 v_1 \cdot v_2 + \lambda_1 v_1 \cdot v_2 \\ &= (\lambda_1 + \lambda_2) v_1 \cdot v_2 . \end{aligned}$$

<u>So</u>: $-2 + -2,\ -2 + 0,\ -2 + 2,$
$0 + 0,\ 0 + 2,\ 2 + 2.$

16.10 = 17.7.

Eigenvalues of $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$



$\bullet$: multiplicity 1
$\odot$: multiplicity 2

Now by general theory (unproved so far)
$\mathfrak{sl}(2)$ is simple (no ideals)
hence semisimple (direct sum of simples)
hence reductive. (take this for granted)

So $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is a direct sum of irreducibles,
and we know all irreducibles are $\mathrm{Sym}^k V$
and can be read off from their eigenvalues.

Here just $\mathrm{Sym}^2(\mathrm{Sym}^2 V) = \mathrm{Sym}^4 V \oplus \underbrace{\mathrm{Sym}^0 V}_{\text{Trivial rep'n}}$.

18.1

Ex. Verify this!

Similarly, look at $\mathrm{Sym}^3(\mathrm{Sym}^2 V)$.
What is its dimension?

In general, $\mathrm{Sym}^k \mathbb{R}^d$ (or $\mathbb{C}^d$) has dimension $\binom{k+d-1}{k-1} = \binom{k+d-1}{d}$

Span: $x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}$ with $\Sigma a_i = k$

"Stars and Bars" $\quad * \; * \,||\, * \; * \,|\, * \,|\, *$

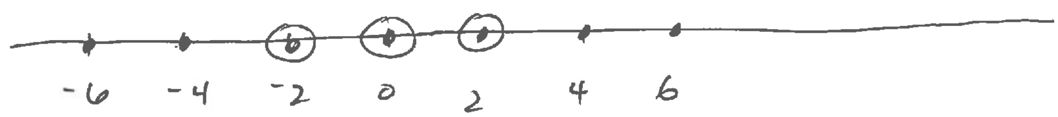
k stars (Here $k=6, d=5$)
$a_1 = 2,\ a_2 = 0,\ a_3 = 2,\ a_4 = 1,\ a_5 = 1$.
$d-1$ bars.

Since $\dim V = 2$, $\dim \mathrm{Sym}^k V = \binom{k+1}{1} = k+1$.

$\dim \mathrm{Sym}^3(\mathbb{C}^3) = \binom{5}{2} = 10$.

17.8 = 18.2

The eigenvalues are $\{-2,0,+2\} + \{-2,0,+2\} + \{-2,0,2\}$.



So $\mathrm{Sym}^3(\mathrm{Sym}^2 V) = \mathrm{Sym}^6 V \oplus \mathrm{Sym}^2 V$.

How to get a map $\mathrm{Sym}^3(\mathrm{Sym}^2 V) \to \mathrm{Sym}^6(V)$?

Multiply all the quadrics.

Some Geometric Plethysm.

You all know what projective space is, right? Good.

Consider the morphism (of projective varieties)

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^n$$

$$(x:y) \longrightarrow (x^n : x^{n-1}y : x^{n-2}y^2 : \cdots : y^n).$$

This is the simplest example of a Veronese embedding

Proposition. Its image is also a variety.

e.g. $\mathbb{P}^1 \xrightarrow{\phi_2} \mathbb{P}^2$

$(x:y) \to (x^2 : xy : y^2)$ $\quad$ $\mathrm{Im}\,\phi = \{(z_0 : z_1 : z_2) : z_0 z_2 - z_1^2 = 0\}$

$\mathbb{P}^1 \xrightarrow{\phi_3} \mathbb{P}^3$

$$\mathrm{Im}\,\phi : \left\{ (z_0 : z_1 : z_2 : z_3) : \begin{array}{l} z_0 z_2 - z_1^2 \\ z_1 z_3 - z_2^2 \\ z_0 z_3 - z_1 z_2 \end{array} \right\}$$

17.9 = 18.3

This is the twisted cubic curve.

Exercise. (1) $\mathrm{Im}\,\phi$ is what I said it was (and not only contained in it)

(2) You really need all three equations (despite being codimension 2 — it is not a "complete intersection")

$$\mathbb{P}^1 \xrightarrow{\phi_n} \mathbb{P}^n$$

$$\mathrm{Im}\,\phi = \left\{ (z_0 : \cdots : z_n) \;:\; \operatorname{rank}\begin{pmatrix} z_0 & z_1 & z_2 & \cdots & z_{n-1} \\ z_1 & z_2 & z_3 & \cdots & z_n \end{pmatrix} = 1 \right\}$$

$$= \{ (z_0 : \cdots : z_n) \;:\; \text{all the } 2\times 2 \text{ minors vanish} \}$$

(so it really is a projective variety!)

Exercise. Prove all this.

Note: This idea generalizes, e.g.

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

$$[x : y : z] \longrightarrow [x^2 : xy : xz : y^2 : yz : z^2]$$

The images are always varieties.
This is cool and well worth learning!

18.4

Fact. $\operatorname{Aut}(\mathbb{P}^n) = PGL_{n+1}(\mathbb{C}) := GL_{n+1}(\mathbb{C}) / \text{scalars}$.

Back to plethysm:

Have the Veronese embedding $\iota_2 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

$[x:y] \to [x^2 : xy : y^2]$

$\operatorname{Im} \iota_2 = \{ \underbrace{[u:v:w]}^{\in \mathbb{P}^2} \mid uw = v^2 \} =: C_2$.

$= V(uw - v^2) \subseteq \mathbb{P}^2$.

SL_2 acts on $\operatorname{Sym}^2 V$ (really $\operatorname{Sym}^2 V^*$ if we're being precise)

and therefore also $\mathbb{P}(\operatorname{Sym}^2 V)$.

What is the action? $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$, etc.

So $x \to ax + cy$, $y \to bx + dy$

$x^2 \to (ax+cy)^2$, $xy \to (ax+cy)(bx+dy)$,

$y^2 \to (bx+dy)^2$

We have by construction

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "g \in SL(2)"] \arrow[d, hook, "\iota_2"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{P}^1 \arrow[d, hook, "\iota_2"] \\ \mathbb{P}^2 \arrow[r, "g"] & \mathbb{P}^2 \end{tikzcd}$$

and so the action of SL_2 on $\operatorname{Sym}^2 V$ takes C_2 to itself.

18.5 Now look at $C_2 \subseteq \mathbb{P}^2(u,v,w)$

SL_2 also acts on $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$, quadratic polynomials on $\mathbb{P}^2$.

This action must preserve the subspace $\mathbb{C}\cdot F$ where $F = uw - v^2$.

There's our trivial subrepresentation!

So we get an ES

$$0 \longrightarrow \underset{\substack{\| \\ \mathbb{C}}}{\mathrm{Sym}^0 V} \longrightarrow \underbrace{\mathrm{Sym}^2(\mathrm{Sym}^2 V)) \longrightarrow \mathrm{Sym}^4 V}_{} \longrightarrow 0$$

Lowbrow description:
Quadratic polynomials in $u = x^2$, $v = xy$, $w = y^2$ are just quartic polynomials in x and y.

Highbrow description:
Pullback via ι_2.

$$\mathbb{P}^1 \xrightarrow{\iota_2} \mathbb{P}^2$$
$$[x:y] \longrightarrow [x^2 : xy : y^2]$$

~~ι_2^* (Quadratic~~

ι_2^* : Quadratic polys on $\mathbb{P}^2$ $\longrightarrow$ Quartic polys on $\mathbb{P}^1$

$$\iota_2^* F(x,y) = F(\iota_2(x,y)).$$

Kernel as above.

Q. Can we describe $\mathrm{Sym}^4(V) \subseteq \mathrm{Sym}^2(\mathrm{Sym}^2 V)$ geometrically?

18.6

Prop. The subrepresentation $\mathrm{Sym}^4(V) \subseteq \mathrm{Sym}^2(\mathrm{Sym}^2 V)$ is the space of conics spanned by the double lines tangent to $C = C_2$.

Proof. Let's see what they are.

A point on C_2 is $[1 : q : q^2]$

$C: uw - v^2 = 0. \quad = \{F(u,v,w) = 0\}$

Tangent line is $\left.\frac{\partial F}{\partial u}\right|_p (u - u_0) + \left.\frac{\partial F}{\partial v}\right|_p (v - v_0) + \left.\frac{\partial F}{\partial w}\right|_p (w - w_0) = 0$

$q^2(u-1) - 2q(v-q) + 1\cdot(w - q^2) = 0$

$q^2 u - 2qv + w + (-q^2 + 2q^2 - q^2) = 0$

So: $q^2 u - 2qv + w = 0$.

The doubled line is $(q^2 u - 2qv + w)^2 = 0$

$q^4 u^2 + q^3(-2uv) + q^2(2uw + 4v^2) + q(-4vw) + w^2$

Let q range over $\mathbb{R}$. What vector space do these span?

$\mathrm{Span}\{u^2, uv, uw+2v^2, vw, w^2\}$.

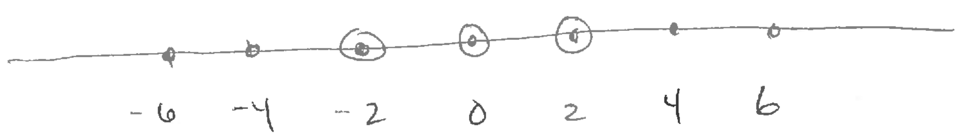
By construction this is invariant under $SL(2)$.

Complementary subspace: Spanned by $F = uw - v^2$ itself.

18.7

More decompositions.

$Sym^3(Sym^2 V)$ EV $\{-2,0,2\} + \{-2,0,2\} + \{-2,0,2\}$



So $Sym^3(Sym^2 V) \cong Sym^6(V) \oplus Sym^2(V)$.

The $Sym^6(V)$: Iso (via i_2^*) to the space of sextic polynomials.

What is $Sym^2(V)$? The space of cubic polynomials on $\mathbb{P}^2$ which vanish on C_2.

This is exactly the kernel of the map

$$Sym^3(Sym^2 V) \twoheadrightarrow Sym^6(V).$$

Since C_2 is quadratic,

Cubic polys which vanish on C_2 is $\operatorname{Span}\{F_x, F_v, F_w\}$.

Yup! That's a $\mathbb{P}^2$ so we see the usual action on Sym^2.

15.8

Other Goodies.

$$\mathrm{Sym}^4(\mathrm{Sym}^2 V) \cong \underbrace{\mathrm{Sym}^8(V)}_{\text{Can see this one!}} \oplus \mathrm{Sym}^4(V) \oplus \mathrm{Sym}^0(V)$$

Vanishing on C modulo F^2.

Spanned by F^2.

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) \cong \mathrm{Sym}^2(\mathrm{Sym}^3 V)$$

and the map $\mathrm{Sym}^2(\mathrm{Sym}^3 V) \to \mathrm{Sym}^2(V)$ is a <u>quadratic</u> map {binary cubic forms} $\to$ {binary quadratic forms}

<u>Exterior</u> <u>Powers</u>:

$\Lambda^2(\mathrm{Sym}^3 V)$

Eigenvalues $\{3, 1, -1, -3\}$ + itself, can't use any of them twice.

$$\Lambda^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^4 V \oplus \mathrm{Sym}^0(V).$$

Interpret that. Etc.

The big questions.

Recall:

(1) Every Lie group G has a Lie algebra $\mathfrak{g}$

(2) A cts hom $\Phi : G \to H$ yields $\phi : \mathfrak{g} \to \mathfrak{h}$

$$\phi(X) = \frac{d}{dt} \Phi(e^{tX})\Big|_{t=0}$$

(3) If $H \leq G$ matrix Lie groups, then $\mathfrak{h} \leq \mathfrak{g}$.

The hard direction. Can we go from the algebra to the group?

Q1. Is every fd real algebra the Lie algebra of some matrix Lie group? (Yes.)

Q2. Given $G, H, \mathfrak{g}, \mathfrak{h}$, $\phi : \mathfrak{g} \to \mathfrak{h}$.
Does there exist a Lie group hom $\Phi : G \to H$ inducing ϕ?
(No, but yes if G is simply connected.)

Q3. If $G, \mathfrak{g}$, $\mathfrak{h} \leq \mathfrak{g}$, is there a matrix Lie group $H \leq G$ with Lie algebra $\mathfrak{h}$? (No, but sort of.)

The Baker - Campbell - Hausdorff formula:
If X and Y are small, then

$$\log(e^X e^Y) = X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}[X, [X, Y]] - \tfrac{1}{12}[Y, [X, Y]] + \cdots$$

In other words,

$$e^X e^Y = \exp(X + Y + \tfrac{1}{2}[X, Y] + \text{etc.})$$

So we know how to multiply any two elements near I in terms of the Lie algebra.

Example. Given $G, H, \mathfrak{g}, \mathfrak{h}, \phi : \mathfrak{g} \to \mathfrak{h}$

Want to construct $\Phi : G \to H$ with $\Phi(e^X) = e^{\phi(X)}$.

We can define a continuous function

$$\begin{tikzcd} U \ni I \arrow[d, phantom, "\cap"] \arrow[dr, "\Phi"] & \\ G \arrow[r, squiggly] & H \end{tikzcd} \quad \text{with} \quad \Phi(A) = e^{\phi(\log A)}$$

because we know that there is a homeomorphism

$$\begin{tikzcd} U \subseteq G \arrow[d, "\log"', bend right] \arrow[d, leftrightarrow] \\ V \subseteq \mathfrak{g} \arrow[u, "\exp"', bend right] \end{tikzcd}$$

So we will get $\Phi(e^X) = e^{\phi(X)}$ for small $X \in \mathfrak{g}$.

But does this have any nice algebraic properties?

e.g. $\Phi(e^X e^Y) = e^{\phi(X)} e^{\phi(Y)}$?

If we believe BCH, then

$$e^X e^Y = \exp\left(X + Y + \tfrac{1}{12}[X,[X,Y]] + \text{etc.}\right) =: e^Z$$

and we will get $\Phi(e^X e^Y) = \Phi(e^Z) = e^{\phi(Z)}$

$$\begin{aligned} \text{But } \phi(Z) &= \phi\left(X + Y + \tfrac{1}{12}[X,[X,Y]] + \cdots\right) \\ &= \phi(X) + \phi(Y) + \tfrac{1}{12}[\phi(X),[\phi(X),\phi(Y)]] + \cdots \\ &= \log\left(e^{\phi(X)} e^{\phi(Y)}\right) \end{aligned}$$

and so $\Phi(e^X e^Y) = e^{\phi(X)} e^{\phi(Y)}$.

So we win! Get a local homomorphism!

Example. Let $G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$ be the Heisenberg group

Its Lie algebra is $\mathfrak{g} = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$.

Then $[\mathfrak{g}, \mathfrak{g}] = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$. (i.e. $\mathfrak{g}$ is nilpotent)

__Special Case of BCH.__

Let $X, Y \in M_n(\mathbb{C})$ with

$$[X, [X, Y]] = [Y, [X, Y]] = 0.$$

(This always happens for the Heisenberg group.)

Then $e^X e^Y = e^{X + Y + \frac{1}{2}[X, Y]}$.

__Proof.__ Will prove, for all $t \in \mathbb{R}$,

$$e^{tX} e^{tY} = \exp\left(tX + tY + \frac{t^2}{2}[X, Y]\right) \quad \text{(so plug in } t=1)$$

$$= \exp(tX + tY) \exp\left(\frac{t^2}{2}[X, Y]\right)$$

(used __hypothesis!!!__)

Or

$$\underbrace{e^{tX} e^{tY} e^{-\frac{t^2}{2}[X, Y]}}_{A(t)} = \underbrace{e^{t(X+Y)}}_{B(t)}$$

19.4

We have $\dfrac{dB}{dt} = B(t)\cdot(X+Y)$.

By the product rule,

These commute.

$$\frac{dA}{dt} = \underbrace{e^{tX} X e^{tY}}\, e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} Y e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}(-t[X,Y])$$

$$Xe^{tY} = e^{tY} e^{-tY} X e^{tY}$$

$$= e^{tY} \mathrm{Ad}_{e^{-tY}}(X)$$

Proved this earlier (3.35)

$$= e^{tY} e^{-t\,\mathrm{ad}_Y}(X).$$

Now $e^{-t\,\mathrm{ad}_Y}(X) = X - t[Y,X] + \underbrace{\frac{t^2}{2}[Y,[Y,X]] + \cdots}_{0}$

So
$$\frac{dA}{dt} = e^{tX} e^{tY}(X - t[Y,X])\, e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} Y e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} \cancel{e^{-\frac{t^2}{2}[X,Y]}}\cdot(-t[X,Y])\, e^{-\frac{t^2}{2}[X,Y]}$$

$$= e^{tX} e^{tY}(X+Y)\, e^{-\frac{t^2}{2}[X,Y]}$$

$$= e^{tX} e^{tY} \cancel{(X+Y)}\, e^{-\frac{t^2}{2}[X,Y]}(X+Y)$$

$$= A(t)(X+Y).$$

So $A(t)$ and $B(t)$ satisfy the same ODE. So we're done.

19.5 So we get:

Thm. Let H be the Heisenberg group, $\mathfrak{h}$ its Lie alg.
G any matrix Lie group w/ alg. $\mathfrak{g}$, $\phi : \mathfrak{h} \to \mathfrak{g}$
$\exists$ a ! Lie gp hom $\Phi : H \to G$ with $\Phi(e^X) = e^{\phi(X)}$

for all $X \in \mathfrak{h}$.

Proof. Follows from the above, plus the fact (for this group) that $\exp : \mathfrak{h} \to H$ is one-to-one and onto.

The formula in general.

$$\log(e^X e^Y) = \log\left(\left(I + X + \frac{X^2}{2} + \cdots\right)\left(I + Y + \frac{Y^2}{2} + \cdots\right)\right)$$

$$= \log\left(I + X + Y + \frac{X^2}{2} + \frac{Y^2}{2} + XY + \cdots\right)$$

$$= X + Y + \frac{X^2}{2} + \frac{Y^2}{2} + XY - \frac{1}{2}(X+Y)^2$$

$$= X + Y + \frac{1}{2}(XY - YX) + \cdots$$

Was it obvious the quadratic term wouldn't have multiples of X^2, Y^2, $XY + YX$, etc.?

Some analysis. In $\{z : |z - 1| < 1\} \subseteq \mathbb{C}$ we have

$$g(z) = \frac{\log z}{1 - \frac{1}{z}} = \sum_{m=0}^{\infty} a_m (z-1)^m$$

You can compute these *if* you want

19.6 (Review.)

Given a vector space V, if $A \in \mathrm{End}(V)$ satisfies $\|A - I\| < 1$, then

$$g(A) = \sum_{m=0}^{\infty} a_m (A - I)^m$$ is defined and convergent.

<u>Baker - Campbell - Hausdorff Theorem.</u>

For all sufficiently small $X, Y \in M_n(\mathbb{C})$ we have

$$\log(e^X e^Y) = X + \int_0^1 g(e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y})(Y)\,dt.$$

What does it mean?

$e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y}$ are linear operators on $M_n(\mathbb{C})$

— hence so is $g(e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y})$. So apply it to Y.

Since X, Y small, $e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y}$ is close to I.

An analytic lemma.

<u>Lemma</u> (Thm 5.4). For $X, Y \in M_n(\mathbb{C})$ we have

$$\frac{d}{dt} e^{X+tY}\Big|_{t=0} = e^X \cdot \left(\frac{I - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X} (Y) \right).$$

What the hell does it mean?

First, notice that $\frac{d}{dt} e^{X+tY}$ is well defined.

As entire functions we have

$$\frac{1 - e^{-z}}{z} = \frac{1}{z}\left[1 - 1 + z - \frac{z^2}{2} + \frac{z^3}{6!} - \cdots\right] = \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{(k+1)!}$$

19.7 $\overset{=20.1}{\text{So,}}$ $\dfrac{I - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}$:= that power series,

$$\frac{I - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}(Y) = Y - \frac{[X,Y]}{2!} + \frac{[X,[X,Y]]}{3!} - \cdots$$

More generally,

Lemma 5.4': $\dfrac{d}{dt} e^{X(t)} = e^{X(t)}\left(\dfrac{I - e^{-\mathrm{ad}_{X(t)}}}{\mathrm{ad}_{X(t)}} \left(\dfrac{dX}{dt} \right) \right).$

Lemma 5.5. If Z is a linear operator on a FD vector space,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \left(e^{-Z/m}\right)^k = \frac{1 - e^{-Z}}{Z}.$$

Proof. If we just had $Z \in \mathbb{C}$, would have

$$\frac{1}{m} \sum_{k=0}^{m-1} \left(e^{-Z/m}\right)^k = \frac{1}{m} \cdot \frac{1 - e^{-Z}}{1 - e^{-Z/m}} \xrightarrow{m\to\infty} \frac{1 - e^{-Z}}{Z}.$$

So you have to make something like this work:

$$\frac{1 - e^{-x}}{x} = \int_0^1 e^{-tx}\, dt \quad \text{for } x \in \mathbb{C}$$

and so

$$\frac{1 - e^{-Z}}{Z} = \underbrace{\int_0^1 e^{-tZ}\, dt}$$

Expand out the def. of e^{-tZ} and integrate term by term.

Now, $\dfrac{1}{m} \sum_{k=0}^{m-1} \left(e^{-Z/m}\right)^k$ is a Riemann sum approximation to the matrix valued integral!

So we win by calculus.

20.2 Proof of Lemma 5.4.

The "~~more generally~~ first part" implies the ~~first part~~ more generally.

Write $\Delta(X,Y) = \frac{d}{dt}\left(e^{X+tY}\right)\Big|_{t=0}$

continuous in X and Y; linear in Y for fixed X.

Apply the product rule to

$$e^{X+tY} = \left[\exp\left(\frac{X}{m} + t\frac{Y}{m}\right)\right]^m.$$

$$\frac{d}{dt}\left(e^{X+tY}\right)\Big|_{t=0} = \sum_{k=0}^{m-1} \left(e^{X/m}\right)^{m-k-1}\left[\frac{d}{dt}\exp\left(\frac{X}{m}+t\frac{Y}{m}\right)\Big|_{t=0}\right]\left(e^{\frac{X}{m}}\right)^k$$

$$= e^{\frac{(m-1)X}{m}} \sum_{k=0}^{m-1} \left(e^{X/m}\right)^{-k} \Delta\left(\frac{X}{m}, \frac{Y}{m}\right)\left(e^{X/m}\right)^k$$

$$\cancel{e^{\frac{(m-1)X}{m}} \frac{1}{m}\sum_{k=0}^{m} \exp\left(-\frac{ad_X}{m}\right)^k \left(\Delta\left(\frac{X}{m}, Y\right)\right)},$$

$$= e^{\frac{m-1}{m}X} \cdot \sum_{k=0}^{m-1} Ad_{\left(e^{X/m}\right)^{-k}}\left(\Delta\left(\frac{X}{m}, \frac{Y}{m}\right)\right)$$

$$= e^{\frac{m-1}{m}X} \sum_{k=0}^{m-1} \exp\left(-\frac{ad_X}{m}\right)^k \left(\Delta\left(\frac{X}{m}, \frac{Y}{m}\right)\right)$$

$$= e^{\frac{m-1}{m}X} \cdot \frac{1}{m} \sum_{k=0}^{m-1} \exp\left(-\frac{ad_X}{m}\right)^k \left(\Delta\left(\frac{X}{m}, Y\right)\right).$$

Now send $m \to \infty$. $e^{\frac{m-1}{m}X} \to X$. $\Delta\left(\frac{X}{m}, Y\right) \to \Delta(0,Y) = Y$.

Get $\lim\limits_{m\to\infty} e^X \cdot \frac{1}{m} \sum_{k=0}^{m-1} \exp\left(-\frac{ad_X}{m}\right)^k Y \underset{\text{Lemma}}{=} e^X \cdot \frac{1-e^{-ad_X}}{ad_X}(Y)$.

20.3 So, to prove BCH, for small X, Y, set

$$Z(t) = \log(e^X e^{tY}) \quad \text{and compute } Z(1).$$

Now, $e^{-Z(t)} \frac{d}{dt} e^{Z(t)} = (e^X e^{tY})^{-1} e^X e^{tY} Y = Y.$

By Thm 5.4,

$$e^{-Z(t)} \frac{d}{dt} e^{Z(t)} = \left(\frac{I - e^{-\mathrm{ad}_{Z(t)}}}{\mathrm{ad}_{Z(t)}} \right) \left(\frac{dZ}{dt} \right)$$

So

$$\frac{dZ}{dt} = \left(\frac{I - e^{-\mathrm{ad}_{Z(t)}}}{\mathrm{ad}_{Z(t)}} \right)^{-1} (Y)$$

in 0 and close enough to identity such that this is invertible

We also have

$$\mathrm{Ad}_{e^{Z(t)}} = \mathrm{Ad}_{e^X} \mathrm{Ad}_{e^{tY}} \quad \text{(by definition)}$$

$$e^{\mathrm{ad}_{Z(t)}} = e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y} \quad \text{Rel'n btn. Ad and ad}$$

$$\mathrm{ad}_{Z(t)} = \log(e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y})$$

So

$$\frac{dZ}{dt} = \left(\frac{I - (e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y})^{-1}}{\log(e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y})} \right)^{-1} (Y)$$

Use our preparatory lemma! $g(z) = \left(\frac{1 - z^{-1}}{\log z} \right)^{-1}$

Get

$$\frac{dZ}{dt} = g(e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y})(Y)$$

Integrate:

$$\underbrace{Z(1) - Z(0)}_{\log(e^X e^Y) - X} = \int_0^1 g(e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y})(Y)\,dt$$

and done!

20.4 .

How to get the series form?

$$g(z) = 1 + \frac{1}{2}(z-1) - \frac{1}{6}(z-1)^2 + \frac{1}{12}(z-1)^3 + \cdots$$

$$e^{ad_X} e^{t\,ad_Y} - I = \left(I + ad_X + \frac{(ad_X)^2}{2} + \cdots\right)\left(I + t\,ad_Y + t^2\frac{(ad_Y)^2}{2} + \cdots\right) - I$$

$$= ad_X + t\,ad_Y + t\,ad_X\,ad_Y + \frac{(ad_X)^2}{2} + t^2\frac{(ad_Y)^2}{2} + \cdots$$

and so

$$g(e^{ad_X} e^{t\,ad_Y})$$

$$= 1 + \frac{1}{2}(\text{above}) + \frac{1}{6}(\text{above})^2 + \cdots$$

= I can compute

$$I + \frac{1}{2}\left(ad_X + t\,ad_Y + t\,ad_X\,ad_Y + \frac{(ad_X)^2}{2} + \frac{t^2(ad_Y)^2}{2}\right)$$

$$- \frac{1}{6}\left((ad_X)^2 + t^2(ad_Y)^2 + t\,ad_X\,ad_Y + t\,ad_Y\,ad_X\right).$$

Also. Since we are applying to Y, $ad_Y(Y) = 0$. Can delete any term with ad_Y acting first.

$$\log(e^X e^Y)$$

$$= X + \int_0^1 \Big(Y + \frac{1}{2}[X,Y] + \frac{1}{4}[X,[X,Y]] - \frac{1}{6}[X,[X,Y]] + \frac{t}{6}[Y,[X,Y]] + \cdots\Big)$$

$$= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots$$

20.5 An alternative proof. (Eichler; Stillwell 7.7)

Let $e^A e^B = e^Z$ with $Z = F_1(A,B) + F_2(A,B) + \cdots$

terms of degree i.

i.e. $F_1(A,B) = A+B$

$F_2(A,B) = \frac{1}{2}[A,B]$ etc.

Call a polynomial $p(A,B,C,\cdots)$ Lie if it is a linear combination of $A, B, C, \cdots$ and their brackets.

i.e. don't need "usual" multiplication.

CBH Theorem. $\forall n$ $F_n(A,B)$ is Lie.

Proof. Induction on n.

Since $(e^A e^B) e^C = e^A (e^B e^C)$

we have

$$(1)\quad \sum_{i=1}^{\infty} F_i\left(\sum_{j=1}^{\infty} F_j(A,B), C\right) = \sum_{i=1}^{\infty} F_i\left(A, \sum_{j=1}^{\infty} F_j(B,C)\right)$$

Assume F_m Lie for $m<n$.

Expand out the degree n terms on both sides.

$F_1(F_n(A,B), C) + F_n(F_1(A,B), C)$

+ Lie poly

$= F_n(A, F_1(B,C)) + F_1(A, F_n(B,C))$ + Lie poly

writing $P_1 \equiv P_2$ if $P_1 - P_2$ is Lie, get

$$(2)\quad F_n(A,B) + F_n(A+B, C) \equiv F_n(A, B+C) + F_n(B,C).$$

20.6

Fact 1. $F_n(rA, sA) = 0$ for scalars r, s and $n \geq 2$.
(The matrices commute.)

Fact 2. $F_n(A, 0) = 0$ by above.

Fact 3. $F_n(rA, rB) = r^n F_n(A, B)$.

(*) Replace C by $-B$ in (2):

$$F_n(A,B) + F_n(A+B, -B) \equiv F_n(A, 0) + F_n(B, -B)$$
$$\equiv 0 \quad \text{(Facts 1-2)}$$

So $F_n(A,B) \equiv -F_n(A+B, -B)$. (3)

(*) Replace A by $-B$ in (2):

$$0 \equiv F_n(-B, B) + F_n(0, C) \equiv F_n(-B, B+C) + F_n(B, C)$$

So $0 \equiv F_n(-B, B+C) + F_n(B, C)$.

(*) Replace B, C by A, B

(4) $F_n(A,B) \equiv -F_n(-A, A+B)$.

(*) $F_n(A, B) \equiv -F_n(-A, A+B)$ (4)

$$\equiv -(-F_n(-A+A+B, -A-B))$$
$$= F_n(B, -A-B)$$
$$\equiv -F_n(-B, -A) \quad (4)$$
$$\equiv -(-1)^n F_n(B, A) \quad \text{homogeneity.}$$

(5) So: $F_n(A,B) = -(-1)^n F_n(B, A)$.

(*) Replace C by $-B/2$ in (2).

$$F_n(A,B) + F_n(A+B, -B/2) \equiv F_n(A, B/2) + F_n(B, -B/2)$$
$$\equiv F_n(A, B/2) \quad \text{(fact 1)}$$

(6) $F_n(A,B) + F_n(A+B, -B/2) \equiv F_n(A, B/2)$.

20.7.

Replace A by $-B/2$ in (2).

$$F_n(-B/2, B) + F_n(B/2, C) \equiv F_n(-B/2, B+C) + F_n(B, C)$$

so $$F_n(B/2, C) \equiv F_n(-B/2, B+C) + F_n(B, C)$$

Replace B, C by A, B:

(7) $$F_n(A, B) \equiv F_n(A/2, B) - F_n(-A/2, A+B).$$

Now use (6) in (7).

$$F_n(A/2, B) \equiv F_n(A/2, B/2) - F_n(A/2 + B, -B/2)$$

$$\equiv F_n(A/2, B/2) + F_n(A/2 + B/2, B/2) \text{ by } (3)$$

$$\equiv 2^{-n}(F_n(A, B) + F_n(A+B, B))$$

$$F_n(-A/2, A+B) \equiv F_n\left(-\frac{A}{2}, \frac{A}{2} + \frac{B}{2}\right) - F_n\left(\frac{A}{2} + B, -\frac{A}{2} - \frac{B}{2}\right) \quad (6)$$

$$\equiv -F_n\left(\frac{A}{2}, \frac{B}{2}\right) + F_n\left(\frac{B}{2}, \frac{A}{2} + \frac{B}{2}\right) \quad (4), (3)$$

$$\equiv -2^{-n}(F_n(A, B) + F_n(B, A+B)).$$

So (7) becomes

* $$F_n(A, B) \equiv 2^{-n}\left[2 \cdot F_n(A, B) + F_n(A+B, B) + F_n(B, A+B)\right]$$

by (5)

(8) $$(1 - 2^{1-n}) F_n(A, B) \equiv 2^{-n}(1 + (-1)^n) F_n(A+B, B).$$

n odd $\Rightarrow$ we win.

n even: $$\underbrace{(1 - 2^{1-n}) F_n(A - B, B)} \equiv 2^{1-n} F_n(A, B)$$

(9) $$\equiv -(1 - 2^{1-n}) F_n(A, -B) \text{ by } (3)$$

20.8

So (10) $$-F_n(A,-B) \equiv \frac{2^{1-n}}{1-2^{1-n}} F_n(A,B)$$

$$-F_n(A,B) \equiv \frac{2^{1-n}}{1-2^{1-n}} F_n(A,-B)$$

$$\equiv -\left(\frac{2^{1-n}}{1-2^{1-n}}\right)^2 F_n(A,B)$$

So $$F_n(A,B)\left(-\left(\frac{2^{1-n}}{1-2^{1-n}}\right)^2 + 1\right) \equiv 0$$

So $F_n(A,B) \equiv 0$ and holy crep this actually worked!

21.1

A Big Theorem.

Let G and H be matrix Lie groups w/ algs $\mathfrak{g}$ and $\mathfrak{h}$.
Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie alg hom.
If G is simply connected, there is a unique Lie group hom $\Phi : G \to H$ with $\Phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{g}$.

Recall simply connected means that any loop can be contracted to a point. Or, $\pi_1(G) = 0$.

The hypothesis is necessary!

Recall we previously had a homorphism $SU(2) \to SO(3)$ which was 2-to-1.

It induced an iso on the Lie algebras.

So, $SU(2)$ is not simply connected.

Cor. 5.7. Given G, H simply connected Lie gps. If $\mathfrak{g} \cong \mathfrak{h}$ then $G \cong H$.

Follows from above.
(Check book to see what this depends on.)

21.2

Def. If G, H matrix Lie groups, a <u>local homomorphism</u> $G \to H$ is a pair (U, f) where:

U is a path connected nbd of the identity in G.

$f : U \to H$ cts map with $f(AB) = f(A) f(B)$ wherever A, B, AB all in U.

<u>The use of BCH</u>:

<u>Proposition 5.9</u>: Let G, H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. $\phi : \mathfrak{g} \to \mathfrak{h}$ Lie alg hom, and define $U_\varepsilon \subseteq G$ by

$$U_\varepsilon = \{ A \in G : \|A - I\| < 1, \ \|\log A\| < \varepsilon \}.$$

Then, for some $\varepsilon > 0$ the map $f : U_\varepsilon \to H$

$$f(A) = e^{\phi(\log A)}$$

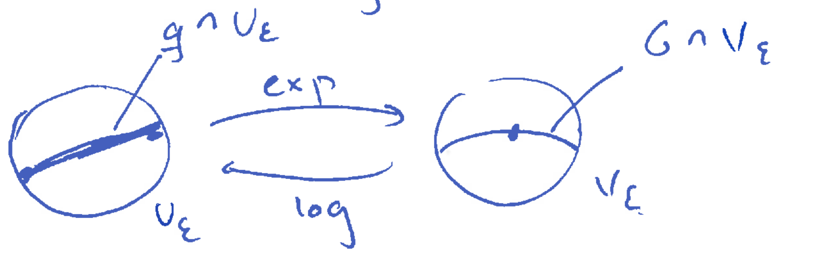
is a local homomorphism.

<u>Proof.</u> Recall

Thm 3.42 Given $G \subseteq GL(n, \mathbb{C})$ with alg $\mathfrak{g}$

$$U_\varepsilon := \{ X \in M_n(\mathbb{C}) : \|X\| < \varepsilon \}$$
$$V_\varepsilon := \exp(U_\varepsilon)$$

Then there exists $\varepsilon \in (0, \log 2)$ such that for $A \in V_\varepsilon$, $A \in G \longleftrightarrow \log A \in \mathfrak{g}$.



Choose ε satisfying this, and such that BCH ~~gives~~ applies to $\log A := X$, $\log B := Y$, $\phi(X)$, $\phi(Y)$ for all $A, B \in U_\varepsilon$.

Now, whenever $AB \in U_\varepsilon$,

$$f(AB) = f(e^X e^Y) = e^{\phi(\log(e^X e^Y))} .$$

Compute that by BCH.

$$\log(e^X e^Y) = X + \int_0^1 a_m \left(e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y} - I\right)^m (Y)\, dt$$

$$\phi\left[\log(e^X e^Y)\right] = \phi(X) + \int_0^1 a_m \left(e^{\mathrm{ad}\,\phi(X)} e^{t\,\mathrm{ad}\,\phi(Y)} - I\right)^m (\phi(Y))\, dt$$

$$= \log\left(e^{\phi(X)} e^{\phi(Y)}\right).$$

(So, we <u>really</u> don't care about the shape of BCH.)

So
$$f(AB) = \exp\left(\phi\left(\log(e^X e^Y)\right)\right)$$
$$= \exp\left(\log\left(e^{\phi(X)} e^{\phi(Y)}\right)\right)$$
$$= f(A) f(B), \quad \text{QED}.$$

<u>Theorem</u> 5.10. Given G, H mat Lie gps with G simply connected. If (U, f) is a local homomorphism $G \to H$, it extends uniquely to a Lie group hom $\Phi : G \to H$.

<u>Step</u> 1. Define it along a path.

<u>Step</u> 2. Prove independence of the path.

21.4

Step 1. Define Φ along a path.

If $A \in G$, there exists some path $A : [0,1] \to G$

$A(0) = I$

$A(1) = A$

Partition $[0,1]$ into $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$

Call this good if whenever s, t in the same interval, $A(t)A(s)^{-1} \in U$.

(Such exists by continuity – see Lemma 3.48.)

Then

$$A = \left(A(1)A(t_{m-1})^{-1}\right)\left(A(t_{m-1})A(t_{m-2})^{-1}\right)\cdots\left(A(t_2)A(t_1)^{-1}\right)A(t_1).$$

Define

$$\Phi(A) = f\left(A(1)A(t_{m-1})^{-1}\right)f\left(A(t_{m-1})A(t_{m-2})^{-1}\right)\cdots f\left(A(t_2)A(t_1)^{-1}\right) f(A(t_1)).$$

Step 2. Independence of the partition.

Prove that if we refine a partition by inserting some s between t_j and t_{j+1}, again get a good partition.

Replace $f(\underbrace{A(t_{j+1})A(t_j)^{-1}})$ by $f(\underbrace{A(t_{j+1})A(s)^{-1}})f(\underbrace{(A(s)A(t_j)^{-1}})$

all these in U.

So these are equal since f is a local homomorphism.

Now, any two partitions have a common refinement

So this gets us independence!

21.5

Step 3. Path independence.

Suppose we have two paths $A_0, A_1 : [0,1] \to G$

with $A_0(0) = A_1(0) = I$

$A_0(1) = A_1(1) = A$.

Then there exists a cts. homotopy

$$A : [0,1] \times [0,1] \longrightarrow G$$

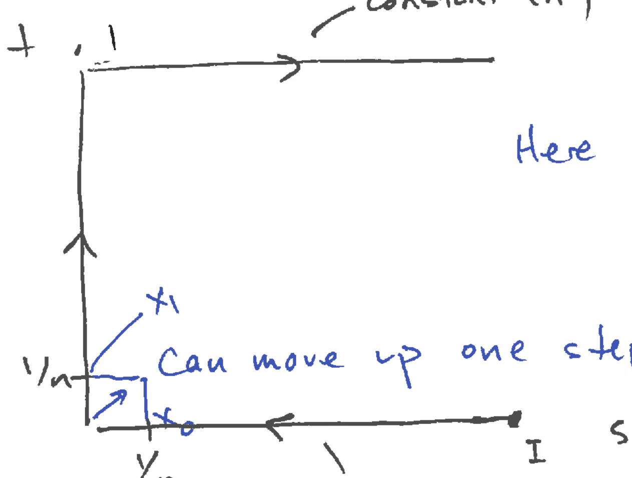
$$A(0,t) = A_0(t)$$
$$A(1,t) = A_1(t)$$
$$A(s,0) = I$$
$$A(s,1) = A \quad \text{(using simple connectedness).}$$

Again by continuity there is N s.t. for $(s,t), (s',t') \in [0,1] \times [0,1]$ with $|s - s'| < \frac{2}{N}$, $|t - t'| < \frac{2}{N}$,

$$A(s,t)\, A(s',t')^{-1} \in U.$$

Deform one path into the other.

Start with this path from $I \to A$



Here ~~A(x)~~ $f(A(x_1)\,A(x_0)^{-1})$
$= f(A(x_1)\,A(x')^{-1})\; f(A(x')\,A(x_0)^{-1})$

Can move up one step at a time as long as these points are all in the small neighborhood.

(Above is just A_0).

Step 4. Φ is a homomorphism agreeing with f on U.

That it's a homomorphism: straight from the definition.

Why does Φ agree with f on U?

Let $A(t)$ be a path joining I to A, choose a good partition for it.

with a different parametrization

Note $t_0, t_1, \dots, t_j$ is a good partition of the path from I to $A(t_j)$.

$$\text{Now } \Phi(A(t_j)) = f(A(t_j)A(t_{j-1})^{-1}) \cdot f(A(t_{j-1})A(t_{j-2})^{-1}) \cdots f(A(t_2)A(t_1)^{-1})\, f(A(t_1))$$

by def.

$$\text{So } \Phi(A(t_1)) = f(A(t_1)).$$

Now, by induction, assume that $\Phi(A(t_j)) = f(A(t_j))$, then

$$\Phi(A(t_{j+1})) = f(A(t_{j+1})A(t_j)^{-1}) \underbrace{f(A(t_j)A(t_{j-1})^{-1}) \cdots f(A(t_1))}$$

$$= \Phi(A(t_j)) = f(A(t_j))$$

$= f(A(t_{j+1}))$ since f is a local homomorphism $A(t_{j+1})A(t_j)^{-1}$, $A(t_j)$ and their product are all in U.

So, by induction $\Phi(A(t_j)) = f(A(t_j))$ for all j and so $\Phi(A) = f(A)$!

21.7

Proof of main theorem

Existence: Given the local homomorphism f of Prop 5.9

$$f : U_\varepsilon \to H$$
$$A \to e^{\phi(\log A)}$$

inducing a global homomorphism Φ.

If $X \in \mathfrak{g}$, then $e^{X/m} \in U$ for m large enough, and

$$\Phi(e^{X/m}) = f(e^{X/m}) = e^{\phi(X)/m}.$$

But now Φ is a homomorphism, so

$$\Phi(e^X) = e^{\phi(X)}$$ and we win.

Uniqueness: Given Φ_1 and Φ_2.

If $A \in G$, write $A = e^{X_1} e^{X_2} \cdots e^{X_N}$ for some $X_j \in \mathfrak{g}$.

Now $\Phi_1(A) = \Phi_1(e^{X_1}) \Phi_1(e^{X_2}) \cdots \Phi_1(e^{X_N})$

$$= e^{\phi(X_1)} e^{\phi(X_2)} \cdots e^{\phi(X_N)} \cdots$$

and the same formula holds for Φ_2.

Last time: (1) G, H mat Lie groups w/ G simply conn'd. If (U, f) is a local hom of G into H, it extends to a Lie group homomorphism $\Phi : G \to H$.

(2) Let G, H mat Lie groups with $\mathfrak{g}$ and $\mathfrak{h}$, let $\phi : \mathfrak{g} \to \mathfrak{h}$ Lie alg hom. Define $U_\varepsilon \subseteq G$ by

$$U_\varepsilon = \{ A \in G : \| A - I \| < 1 \quad \| \log A \| < \varepsilon \}$$

Then there exists $\varepsilon > 0$ s.t.

$$\begin{aligned} U_\varepsilon &\xrightarrow{f} H \\ A &\longrightarrow e^{\phi(\log A)} \end{aligned}$$

is a local homeomorphism.

Now use (1) $\Longrightarrow$

Thm 5.6: If G is simply conn'd, $\exists$ a unique Lie gp hom for $\phi : \mathfrak{g} \to \mathfrak{h}$, $\Phi : G \to H$ s.t. $\Phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{g}$.

Proof. We constructed a local, and hence a global hom $\Phi : G \to H$. Why is it what we said?

If $X \in \mathfrak{g}$, then $e^{X/m} \in U$ for m large enough, and

$$\Phi(e^{X/m}) = f(e^{X/m}) = e^{\phi(X)/m}.$$

So, since Φ is a homomorphism, we win.

$$\Phi(e^X) = e^{\phi(X)}.$$

Why is it unique?

If $A \in G$, can write $A = e^{X_1} \cdots e^{X_N}$ with $X_i \in \mathfrak{g}$, and

$$\Phi_0(A) = \Phi(e^{X_1}) \cdots \Phi(e^{X_N})$$
$$= e^{\phi(X_1)} \cdots e^{\phi(X_n)}.$$

So we don't have any choice what Φ is!

Corollary 5.7 Let G, H simply connected, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.

If $\mathfrak{g} \cong \mathfrak{h}$ then $G \cong H$.

Proof. $\phi : \mathfrak{g} \to \mathfrak{h}$ induces $\Phi : G \to H$

$\psi = \phi^{-1} : \mathfrak{h} \to \mathfrak{g}$ induces $\Psi : H \to G$

Associated to $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity homomorphisms $\mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{h} \to \mathfrak{h}$.

So they must both be the identity.

Universal covers.

If G is a ~~simply~~ connected matrix Lie group, then a universal cover is a simply connected matrix Lie gp H, with a homomorphism $\Phi : H \to G$ inducing an isomorphism $\phi : \mathfrak{h} \to \mathfrak{g}$ of Lie algebras.

Φ is the covering map.

Prop.

Given two universal covers of G

$$\begin{tikzcd} H_1 \arrow[rr, dashed, "\exists!"] \arrow[dr, "\Phi_1"'] & & H_2 \arrow[dl, "\Phi_2"] \\ & G & \end{tikzcd}$$

there exists a unique Lie group iso $\Phi : H_1 \to H_2$ with $\Phi_2 \circ \Phi = \Phi_1$.

So in other words it is a covering space with "the same" Lie algebra.

Corollary. Given

$$\begin{array}{ccc} \tilde{G} & & \\ \downarrow & \text{universal cover} & \\ G & & H \\ \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{h} \end{array}$$

Then there is a unique hom $\Phi : \tilde{G} \to H$ with $\Phi(e^X) = e^{\phi(X)}$ for all $X \in \mathfrak{g}$.

Example. The universal cover of $SO(3)$ is $SU(2)$,

Proof. Recall, we had:

a 2-1 surjection $SU(2) \to SO(3)$

showed $SU(2)$ was simply connected

an isomorphism of Lie algebras.

So we're done ~!

22.4

Proposition. (not proved here)

Every Lie group G has a universal cover. But it may not be a matrix Lie group, even if G is.

Example. $SL(2, \mathbb{R})$.

Theorem. (Polar Decomposition; Ch. 2.5)

Every $A \in SL_n(\mathbb{R})$ can be written uniquely as

$$A = Re^X$$

where $R \in SO(n)$ and X is real, symmetric, with trace zero.

For $n = 2$. $SO(2) = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}$

$X \in W = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix} : a, b \in \mathbb{R} \right\}$.

Kind of like writing $z \in \mathbb{C}$ as $re^{i\theta}$.

So as a manifold, $SL(2) = SO(2) \times W = SO(2) \times \mathbb{R}^2$, which is not simply connected.

Theorem. The universal cover of $SL(2,\mathbb{R})$ is not a matrix Lie group.

More specifically.

Theorem. Let $G \subseteq GL(n,\mathbb{C})$ be conn'd, matrix Lie gp,

$\Phi : G \to SL(2,\mathbb{R})$ Lie group hom

$\phi : \mathfrak{g} \to sl(2,\mathbb{R})$ isomorphism.

Then Φ is an isomorphism too.

In particular G is not simply conn'd so cannot be the universal cover.

Lemma. Suppose $\psi : sl(2,\mathbb{R}) \to gl(n,\mathbb{C})$ Lie alg hom.

Then it extends to a Lie group hom $\Phi : SL(2,\mathbb{R}) \to GL(n,\mathbb{C})$

with $\Phi(e^X) = e^{\phi(X)}$ for all $X \in sl(2,\mathbb{R})$.

(Holds even though $SL(2,\mathbb{R})$ is not simply connected.)

Proof of lemma.

Extend to a map $\psi_{\mathbb{C}} : \underbrace{sl(2,\mathbb{C})}_{\substack{\shortparallel \\ sl(2,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}}} \to gl(n,\mathbb{C})$.

$\psi_{\mathbb{C}} = \psi$ on $sl(2,\mathbb{R})$ and then extend by complex linearity. (Exercise: check this makes sense)

Now $SL(2,\mathbb{C})$ is simply connected.

So get $\Phi_{\mathbb{C}} : SL(2,\mathbb{C}) \to GL(n,\mathbb{C})$

$\Phi_{\mathbb{C}}(e^X) = e^{\psi_{\mathbb{C}}(X)}$ for $X \in sl(2,\mathbb{C})$.

Restrict to $SL_2(\mathbb{R})$!

Proof of theorem.

$$\Phi : G \to SL(2,\mathbb{R})$$

$$\phi : \mathfrak{g} \xrightarrow{\sim} \mathfrak{sl}(2,\mathbb{R})$$

Let $\psi = \phi^{-1}$. It induces (by the lemma) a map $\Psi : SL(2,\mathbb{R}) \to G$ corresponding to ψ.

Since ϕ and ψ are inverses, so must be Ψ and Φ.

What the book doesn't do.

So what is the universal cover of $SL(2,\mathbb{R})$?

The "metaplectic group" $Mp(2,\mathbb{R})$ (a double cover)

Let $SL(2,\mathbb{R})$ act on $\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

$$Mp(2,\mathbb{R}) := \{ (g, \varepsilon) : g \in SL(2,\mathbb{R}),\ \varepsilon : \text{holo fn. on } \mathbb{H} \text{ with } \varepsilon(z)^2 = j(g,z) := cz+d \}$$

Multiplication.

$$(g_1, \varepsilon_1) \cdot (g_2, \varepsilon_2) = (g_1 g_2,\ z \to \varepsilon_1(g_2 \cdot z)\, \varepsilon_2(z))$$

Well defined by the "cocycle condition"

Oops, this is WRONG, it is only a double cover of $SL_2(\mathbb{R})$ but since $\pi_1(SL_2(\mathbb{R})) = \pi_1(SO_2(\mathbb{R})) = \mathbb{Z}$ we need a $\mathbb{Z}$-cover.

23.1

Subgroups and subalgebras.

Q. If G is a matrix Lie gp w/ alg. $\mathfrak{g}$, and $\mathfrak{h} \leq \mathfrak{g}$ subalgebra, does there exist a matrix Lie group $H \leq G$ whose algebra is $\mathfrak{h}$?

Answer. No. Let $G = GL_2(\mathbb{C})$.

$\mathfrak{h} \leq \mathfrak{g}$ $\quad \mathfrak{h} = \left\{ \begin{pmatrix} it & \\ & ita \end{pmatrix} : t \in \mathbb{R} \right\}$

for some irrational a.

Suppose this was the Lie algebra of some Lie gp H.

Then H contains $\left\{ \begin{pmatrix} e^{it} & \\ & e^{ita} \end{pmatrix} : t \in \mathbb{R} \right\}$

and hence its closure

$$H_1 := \left\{ \begin{pmatrix} e^{it} & \\ & e^{i\phi} \end{pmatrix} : t, \phi \in \mathbb{R} \right\}$$

But $\mathfrak{h}$ would have to contain $\left\{ \begin{pmatrix} it & \\ & i\phi \end{pmatrix} : t, \phi \in \mathbb{R} \right\}$.

The problem is "non-local" and topological.

Def. Let H be any subgroup of $GL_n(\mathbb{C})$. Then its Lie algebra $\mathfrak{h}$ is

$$\mathfrak{h} = \{ X \in \mathfrak{gl}_n(\mathbb{C}) : e^{tX} \in H \text{ for all } t \in \mathbb{R} \}.$$

This is a Lie algebra. (Proof omitted)

23.X

Def. Given $G, \mathfrak{g}$, $H \leq G$. If H is a subgroup, we say it is a <u>connected Lie subgroup</u> (analytic subgroup) of G if

(1) Its Lie algebra $\underline{h}$ is a Lie subalg. of $\mathfrak{g}$.
(2) Every elt. of H can be written as $e^{X_1} \cdots e^{X_m}$ with $X_1, \dots, X_m \in \underline{h}$.

Example. $\left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix} : t \in \mathbb{R} \right\}$ is one, even though it's not closed.

<u>Note</u>. Any such is path connected. (Do you see it?)

<u>Theorem</u>. Let G be a matrix Lie gp w/ alg. $\mathfrak{g}$.
$\underline{h} \subseteq \mathfrak{g}$ Lie subalg.
There exists a unique <u>connected Lie subgroup</u> $H \leq G$ with Lie algebra $\underline{h}$.

We can also define a nice topology on it.

To prove this. WLOG $G = GL(n, \mathbb{C})$.
Let $H = \{ e^{X_1} \cdots e^{X_N} : X_1, \dots, X_N \in \underline{h} \}$.
We want to show $\mathrm{Lie}(H) = \underline{h}$.
(Clearly $\mathrm{Lie}(H)$ <u>contains</u> $\underline{h}$.)
If we prove this, then we're done.

23.?

We had a proof like this before.

Think of $gl(n, \mathbb{C}^2) = \mathbb{R}^{2n^2}$

$$gl(n, \mathbb{C}^2) = \underline{h} \oplus D \quad \text{as vector spaces}$$

D : orthogonal complement of $\underline{h}$ wrt usual inner product.

Was proved before: (Thm 3.42) compute derivatives and use inverse function theorem

There exist neighborhoods U and V of 0 in $\underline{h}$, D and a neighborhood $W \ni I$ in $GL(n, \mathbb{C})$ such that every $A \in W$ can be written uniquely as

$$A = e^X e^Y, \quad X \in U, Y \in V$$

with X, Y depending continuously on A.

i.e. something of the direct sum decomposition transfers globally to G.

We do know, by BCH: if X_1 and X_2 are small and in $\underline{h}$, then $e^{X_1} e^{X_2} = e^{X_3}$ with X_3 also in $\underline{h}$.

So locally everything is good.

How do we know we can't make a tangent curve in a different direction by cobbling together a bunch of small things?

Lemma. Define D, V as above, $E \subseteq V$ by

$$E = \{ Y \in V : e^Y \in H \}.$$

Then E is at most countable.

(So you can't make a curve out of it.)

Formal proof of this idea.

(i.e. proof of theorem, assuming lemma)

Let $\mathfrak{h}' = \mathrm{Lie}(H)$ with $\mathfrak{h} \subseteq \mathfrak{h}'$.

If $Z \in \mathfrak{h}'$, write for small t

$$e^{tZ} = e^{X(t)} e^{Y(t)},$$

$$\left.\begin{array}{l} X(t) \in U \subseteq \mathfrak{h} \\ Y(t) \in V \subseteq D \end{array}\right\} \text{ both continuous functions of } t.$$

Since $Z \in \mathrm{Lie}(H)$, $e^{tZ} \in H$ for all t.

Since also $e^{X(t)} \in H$, $e^{Y(t)} \in H$ for all small t.

If $Y(t)$ not constant, it takes on uncountably many values, so E of the lemma is uncountable (contradiction).

So $Y(t)$ is constant, hence identically zero.

So for small t, $e^{tZ} = e^{X(t)}$ so $tZ = X(t) \in \mathfrak{h}$.

So $Z \in \mathfrak{h}$ and $\mathfrak{h}' \subseteq \mathfrak{h}$ QED.

Another lemma.

Pick (arbitrarily!) a basis for $\mathfrak{h}$. Call an element of $\mathfrak{h}$ rational if it is a $\mathbb{Q}$-linear combo of elts. of this basis.

For every $\delta > 0$ and all $A \in H$, there exist rational $R_1, \ldots, R_m \in \mathfrak{h}$ s.t.

$$A = e^{R_1} e^{R_2} \cdots e^{R_m} e^{X} \quad \text{with } X \in \mathfrak{h},\ \|X\| < \delta.$$

(Vague analogue of: $\mathbb{Q}$ is dense in $\mathbb{R}$.)

23.5

Idea. If a δ-ball is in U,
there are countably many m-tuples $(R_1, \dots, R_m)$
$\to$ so H is covered by countably many translates of e^U.

Proof. Choose $\varepsilon > 0$ s.t. BCH holds for $X, Y \in \underline{h}$,
with $\|X\| < \varepsilon$, $\|Y\| < \varepsilon$.

Write $e^X e^Y = e^{C(X,Y)}$ with such X, Y.

$C(X,Y)$ continuous, and can assume (by shrinking δ)
$$\|X\|, \|Y\| < \delta \Rightarrow \|C(X,Y)\| < \varepsilon.$$

Can write any $A \in H$ as
$$A = e^{X_1} \cdots e^{X_N} \quad \text{with } X_j \in \underline{h},\ \|X_j\| < \delta.$$

Induct on N.

if $A = e^{X_1} \cdots e^{X_{N+1}}$

by ind. hypothesis $A = e^{R_1} \cdots e^{R_m} e^{X} e^{X_{N+1}}$ (R_i rational; $X \in \underline{h}$ and $\|X\| < \delta$)
$$= e^{R_1} \cdots e^{R_m} e^{C(X, X_{N+1})}$$

where again $C(X, X_{N+1}) \in \underline{h}$
$\|C(X, X_{N+1})\| < \varepsilon$
(but maybe not δ).

Choose rational $R_{m+1} \in \underline{h}$ (can do it)!
very close to $C(X, X_{N+1})$ with again $\|R_{m+1}\| < \varepsilon$.

So, $A = e^{R_1} \cdots e^{R_m} e^{-R_{m+1}} e^{C(X, X_{N+1})}$
$$= e^{R_1} \cdots e^{R_m} e^{C(-R_{m+1}, C(X, X_{N+1}))}$$

with: $C(-R_{m+1}, C(X, X_{N+1})) \in \underline{h}$ and $\|\cdot\| < \delta$ for "very close" close enough.

Done

23.6

Proof of countability lemma. (and hence we're done.)

Choose δ s.t. $\|X\|, \|Y\| < \delta \Rightarrow C(X,Y)$ defined and in U.

Claim. For each seq of rat'l elts $R_1, \dots, R_m \in \underline{h}$ there is <u>at most one</u> $X \in \underline{h}$ with $\|X\| < \delta$ and

$$(*) \qquad e^{R_1} \cdots e^{R_m} e^{X} \in e^{V}.$$

Proof: if there are two, get e^{Y_1} and $e^{Y_2} \in e^V$ so that $e^{Y_1} e^{-X_1} = e^{Y_2} e^{-X_2}$

$$e^{-Y_1} = e^{-X_1} e^{X_2} e^{-Y_2}$$

$$= e^{C(-X_1, X_2)} e^{-Y_2}$$

with $C(-X_1, X_2) \in U$.

But by hypothesis, each elt. of $e^U e^V$ can be <u>uniquely</u> represented as such.

So $Y_1 = Y_2$, $C(-X_1, X_2) = 0$, $X_1 = X_2$.

Oh, but now we're done.

By lemma, write any $A \in H$ like $(*)$.
For each of countably many seqs $(R_1 \cdots R_m)$ get at most one element of e^V. So we're done!

Theorem. Let H be a conn'd Lie subgroup of $GL(n, \mathbb{C})$ with Lie algebra $\underline{h}$.

Then H can be given the structure of a smooth manifold s.t. the group operations on H are smooth, inclusion $H \hookrightarrow GL(n, \mathbb{C})$ is smooth.

Recall, $\left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix} : t \in \mathbb{R} \right\}$ for your favorite irrational a.

Not the Euclidean topology.

Want the topology from $\mathbb{R}$ here.

Sketch. For $A \in H$, $\varepsilon > 0$,

$U_{A,\varepsilon} = \{ Ae^X : X \in \underline{h} \text{ and } \|X\| < \varepsilon \}$.

Define $U \subseteq H$ to be open if for each $A \in U$, there is ~~A~~ $\varepsilon > 0$ with $U_{A,\varepsilon} \subseteq U$.

IOW: Take these as the basis for a topology.

This is finer than the topology H inherits from G.

A, B close here $\Longrightarrow$ close in ordinary topology in G

open here $\Longleftarrow$ Open in ordinary

(Full proof omitted. See book.)

23.8

Finally (proofs out of scope).

<u>Thm.</u> If $\mathfrak{g}$ is a f.d. real Lie algebra, there is a matrix Lie group with that Lie algebra.

If you're happy with a <u>connected Lie subgroup</u> of $GL(n, \mathbb{C})$:

<u>Proof.</u> (1) By structure theory of Lie algebras, $\mathfrak{g}$ is a real subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ for some n.
Now use theorem.

//

Homework: Ch. 5, 1–4, 6, 8, 9, 14

Representation theory of $sl(3,\mathbb{C})$.

(Hall, Ch. 6 or Fulton–Harris, Ch. 12)

Recap of $sl(2,\mathbb{C})$.

Since $SL(2,\mathbb{C})$ is simply connected, any rep'n of $SL(2)$

$$\Phi : SL(2,\mathbb{C}) \longrightarrow GL(V)$$

is detemined by the rep'n

$$\phi : sl(2,\mathbb{C}) \longrightarrow gl(V)$$

with $\Phi(e^X) = e^{\phi(X)}$.

Φ is irred iff ϕ is.

Now $SL(n)$ and $GL(n)$ are <u>reductive</u>:

Every rep'n is iso to a direct sum of irreducibles.

<u>So</u>: If we want to understand the rep theory of $SL(2)$, enough to understand the <u>ireps</u> of $sl(2)$.

In fact, ~~there~~ they're all $\mathrm{Sym}^k(2)$.

How did we classify?

$sl_2(\mathbb{C})$: span (H, X, Y)

$$[H,X] = 2X \qquad [H,Y] = -2Y$$

$$[X,Y] = H.$$

so ad_H acting on $sl(2)$ is diagonalizable.

24.2

Fact. (Follows from Jordan form) If V is an irrep of $sl(2)$:

The action of H on V (i.e. $H(v) = \pi(H)v$) is diagonalizable.

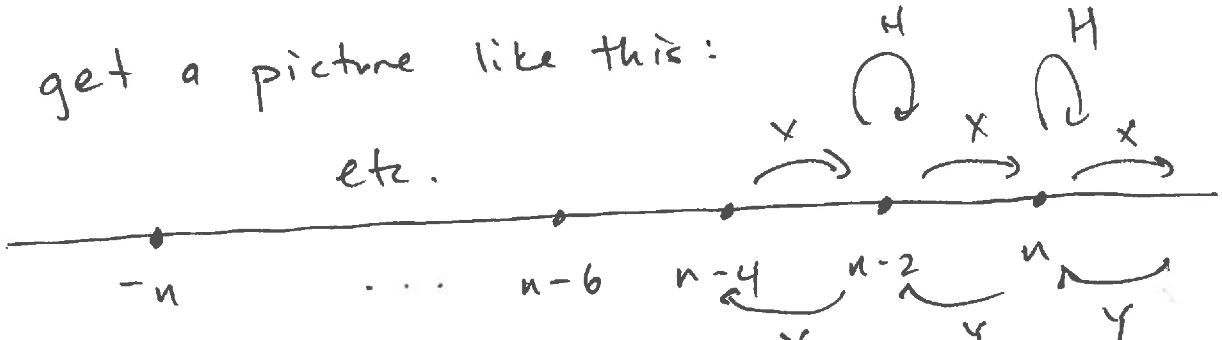
(Last time we just used: it has one EV.)

We have $V = \bigoplus V_\alpha$ with $H(v) = \alpha \cdot v$.

We computed:
$$H(X(v)) = (\alpha + 2)\, X(v)$$
$$H(Y(v)) = (\alpha - 2)\, Y(v)$$

So X sends α-eigenvectors to $(\alpha+2)$-eigenvectors
Y to $(\alpha-2)$-eigenvectors.

We get a picture like this:



and this information detemines the representation.

$\underline{\underline{sl(3)}}$: (Given an irrep V of $sl(3)$.)
In place of H we will define

$$\underline{h} = \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$$

a 2-dimensional subspace of $sl(3)$.

Def. An eigenvector for $\underline{h}$ is a vector $v \in V$ with
$$H(v) = \alpha(H) \cdot v$$
for all $H \in \underline{h}$. Here $\alpha(H)$ will depend linearly on H and so $\alpha \in \mathrm{Hom}(\underline{h}, \mathbb{C}) = \underline{h}^*$.

24.3

Linear Algebra Fact.

Any fd rep'n V of $sl_3(\mathbb{C})$ has a decomposition

$$V = \bigoplus_\alpha V_\alpha$$

where V_α is an eigenspace for $\underline{h}$ and α ranges over a finite subset of $\underline{h}^*$.

Here we call α a ~~weight~~ weight.

We also refer to roots: nonzero weights of the adjoint representation.

e.g. writing $H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$,

$(a_1, a_2) \in \mathbb{C}^2$ is a root if $(a_1, a_2) \neq (0,0)$

and $[H_1, Z] = a_1 Z$, $[H_2, Z] = a_2 Z$

eigenvectors for adjoint action of $\underline{h}$ on $sl_3(\mathbb{C})$. $\longrightarrow$ for some $Z \in sl(3, \mathbb{C})$. (Z is a root vector)

We can equivalently regard the roots as living in $\underline{h}^*$. Writing them in $\mathbb{C}^2$ is possible when we choose a basis for $\underline{h}^*$ (equivalently, $\underline{h}$).

We apply our linear algebra fact to the adjoint action of $sl(3)$ on itself:

$$sl_3(\mathbb{C}) = \underline{h} \oplus \Big(\bigoplus_{\alpha \in \underline{h}^*} \underset{\uparrow\ \text{root vectors, i.e. eigenvectors for } \underline{h}}{\underline{g}_\alpha} \Big)$$

If $H \in \underline{h}$, $Y \in \underline{g}_\alpha$,

$[H, Y] = ad(H)(Y) = \alpha(H) Y$.

Now, write $X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$Y_i = X_i^T \quad (i = 1, 2, 3).$$

Turns out these are all root vectors.

e.g. $[H_1, X_1] = 2X_1$, $[H_2, X_1] = -X_1$ (do on board)

So the associated root is $(2, -1)$

or more precisely the functional $\underline{h}^* \to \mathbb{C}$
$H_1 \to 2$
$H_2 \to -1$
and extend by linearity.

Can check:

		EV for ad(H_1)	EV for ad(H_2)
Root vector:	X_1	2	-1
	X_2	-1	2
	X_3	1	1
	Y_1	-2	1
	Y_2	1	-2
	Y_3	-1	-1

24.5

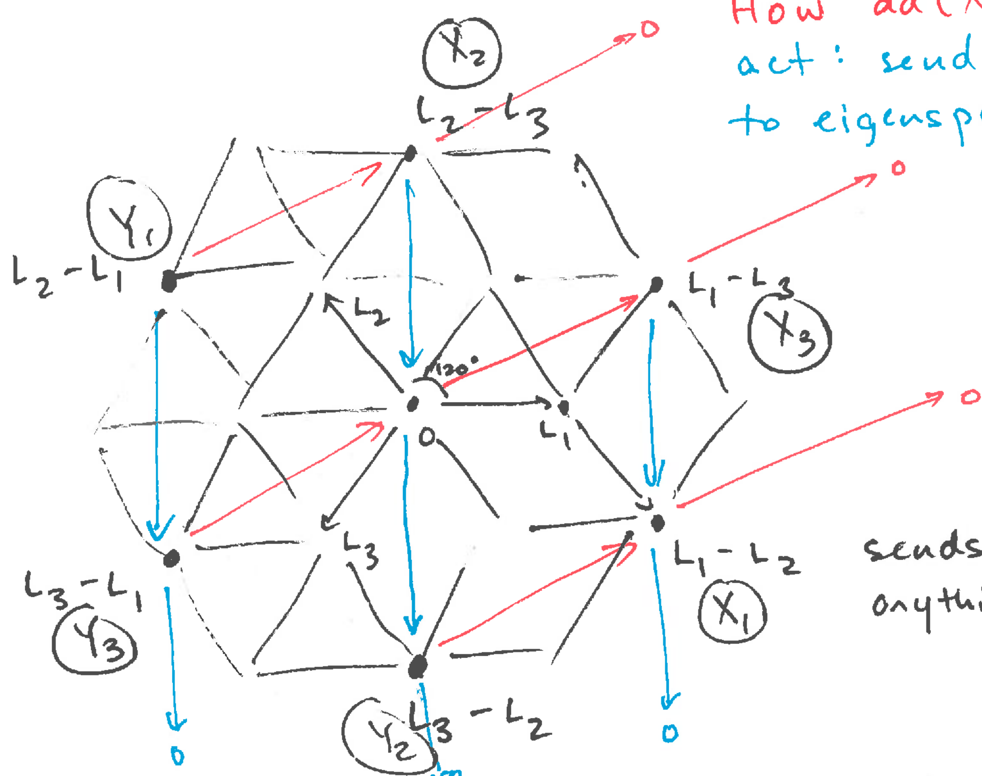
Picture. Not exactly the one you were expecting.

Define L_1, L_2, L_3 : functionals (diag matrices) $\to \mathbb{C}$

$$L_i \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_i$$

so that $\quad h^* = \mathbb{C}\{L_1, L_2, L_3\} / L_1 + L_2 + L_3 = 0.$

How $ad(X_3)$, $ad(Y_2)$ act: send eigenspaces to eigenspaces!



sends $H_1 \to 1$

anything orthogonal to 0.

The $X_1, X_2, X_3, Y_1, Y_2, Y_3$ have these eigenvalues.

e.g.: $[H_1, X_1] = 2X_1 \quad [H_2, X_1] = -X_1$.

$X_1 \in g_\phi$ with $\phi: H_1 \to 2$

$\qquad\qquad\qquad\quad H_2 \to -1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to 2$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \to -1$$

This is the functional $\underline{L_1 - L_2}$.

$X_2 \in \mathfrak{g}_\alpha$ with $H_1 \to -1$, $H_2 \to 2$

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \longrightarrow -1$$

$$\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \longrightarrow 2.$$

This is $\underline{L_2 - L_3}$

$X_3 \in \mathfrak{g}_\alpha$ with $H_1 \to 1$, $H_2 \to 1$

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \to 1$$

$$\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \to 1$$

$\underline{L_1 - L_3}$.

<u>and, if</u> $X_i \in \mathfrak{g}_\alpha$ then $Y_i \in \mathfrak{g}_{-\alpha}$.

24.7

Do the following ~~picture~~ computation. (This is for the adjoint rep'n)

Let $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$.

(i.e. $[H, X] = \alpha(H)\, X$, $[H, Y] = \beta(H)\, Y$ for all $H \in \underline{h}$.

What does ad (H) do to $[X, Y]$?

$$[H, [X, Y]] = [X, [H, Y]] + [[H, X], Y] \quad \text{(Jacobi)}$$
$$= [X, \beta(H)\cdot Y] + [\alpha(H)\cdot X, Y] \quad \text{(above)}$$
$$= (\alpha(H) + \beta(H))\, [X, Y].$$

So $[X, Y]$ is again an eigenvector for $\underline{h}$ with eigenvalue (weight) $\alpha + \beta$.

(Recall again $\underline{h}$ is a 2-dimensional vector space and $\alpha + \beta \in \underline{h}^*$.)

<u>IOW:</u> ad $(\mathfrak{g}_\alpha)$ sends $\mathfrak{g}_\beta \longrightarrow \mathfrak{g}_{\alpha+\beta}$.

So we can see how ad$(\mathfrak{g}_\alpha)$ acts on all the root spaces.

(Note that $[H_1, H_2] = 0$ so $\underline{h}$ itself has weight $(0,0)$.

See the old picture for the action of ad (X_3)

ad (Y_2).

But the same is true of any rep'n V.

Have $V = \bigoplus V_\alpha$, and

whenever $X \in \mathfrak{g}_\alpha \subseteq sl_3(\mathbb{C})$,

$v \in V_\beta \subseteq V$, $H \in \mathfrak{h}$

$$
\begin{aligned}
H(X(v)) &= X(H(v)) + [H, X](v) \\
&= X(\beta(H) \cdot v) + (\alpha(H) \cdot X)(v) \\
&= (\alpha(H) + \beta(H)) \cdot X(v).
\end{aligned}
$$

So if v is an $\mathfrak{h}$-eigenvector with EV α then $X(v)$ is, with EV $\alpha + \beta$.

You can draw the pretty picture again:

The eigenvalues in an irrep of $sl(3, \mathbb{C})$ differ from each other by <u>integral</u> linear combinations of vectors $L_i - L_j \in \mathfrak{h}^*$. (i.e. the roots).

So the adjoint representation determines the rest of the story.

25.1.

Representations of $sl(3) = g$.

Step 1.

We had $\underline{h} = \text{Span}\left(\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}\right)$

with $[\underline{h}, \underline{h}] = 0$

and such that ad_H is diagonalizable.

There is a basis for g s.t. $ad_{\underline{h}}$ acts diagonally for all $\underline{h} \in H$.

This is called a Cartan subalgebra.

Step 2. Let $\underline{h}$ act on g by the adjoint representation, and decompose (a Cartan decomposition)

$$g = \underline{h} \oplus \left(\bigoplus_{\alpha} g_\alpha\right)$$

Each g_α is an eigenvector for H, i.e. we have

$$ad(H)(X) = \alpha(H) \cdot X$$

with $\alpha \in \underline{h}^*$ a functional.

The α are the weights of the adjoint representation

the roots.

we had a pretty picture:

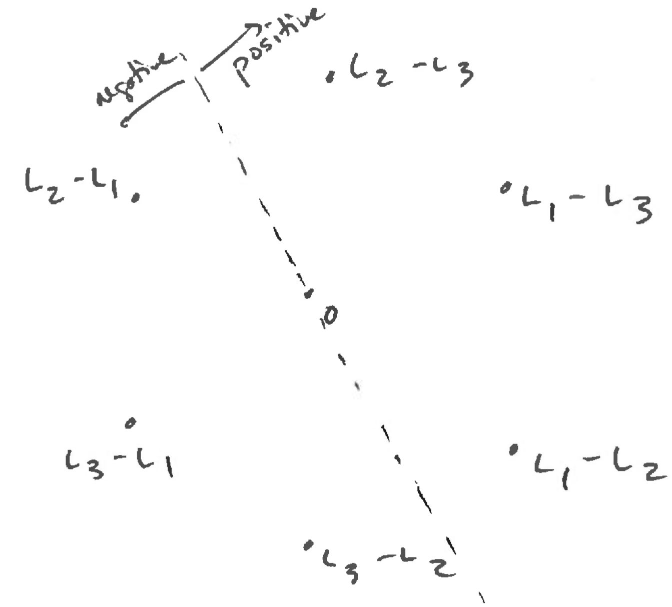
$$\underline{h} = \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$$

Define functionals $L_i : \underline{h} \to \mathbb{C}$ read off ith coordinate.

Then

$$\underline{h}^* = \mathbb{C}\{L_1, L_2, L_3\} / L_1 + L_2 + L_3 = 0$$

25.2. The picture of the root system.



Recall, $\mathrm{ad}(\mathfrak{g}_\alpha) : \mathfrak{g}_\beta \longrightarrow \mathfrak{g}_{\alpha+\beta}$.

Now, if V is <u>any</u> f.d. rep'n of $\mathfrak{g}$, again have $V = \oplus V_\alpha$

(eigenspaces for <u>action</u> of $\mathfrak{h}$ - of course it may not be adjoint!)

and if $X \in \mathfrak{g}_\alpha$ and $v \in V_\beta$,

$$H(X(v)) = X(H(v)) + [H, X](v)$$
$$= X(\beta(H) \cdot v) + (\alpha(H) \cdot X)(v)$$
$$= (\alpha(H) + \beta(H))\, X(v).$$

So same thing!

The action of $\mathfrak{g}_\alpha$ carries <u>V_β to $V_{\alpha+\beta}$</u>.

So the eigenvalues differ by the roots.

25.3.

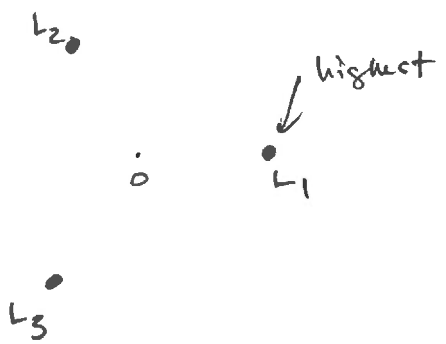
Define the positive roots to be $L_1 - L_2$, $L_1 - L_3$, $L_2 - L_3$.
(this is somewhat arbitrary)

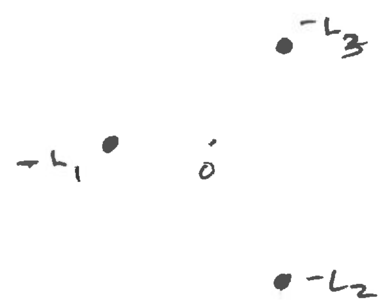
Definition. A vector $v \in V$ is called a highest weight vector if it is in some V_α and it is killed by all the positive roots.

(Lemma. One exists)

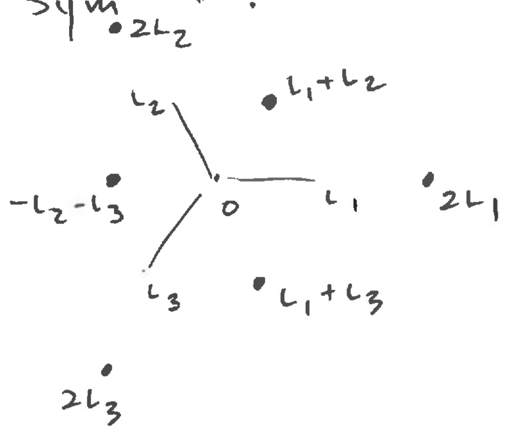
Examples. (More plethysm!!)

Standard rep'n $V \cong \mathbb{C}^3$, Dual rep'n V^* [my pictures suck ^^]

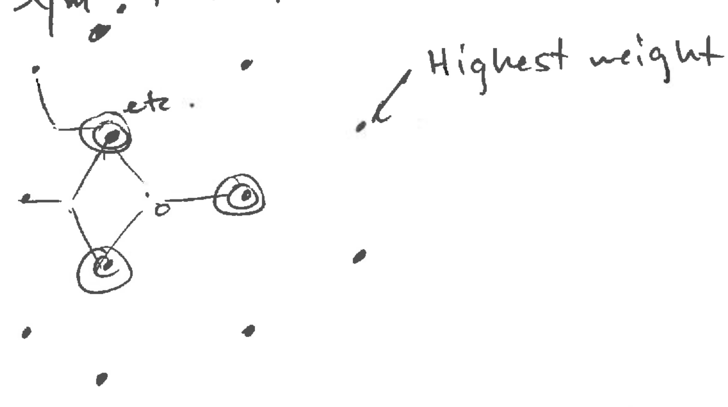




$\mathrm{Sym}^2 V$.



$\mathrm{Sym}^2 V \otimes V^*$



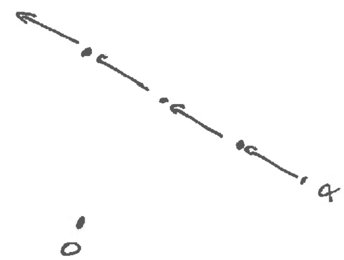
This one is not irreducible!

So how to find every irreducible rep'n.

Find a highest weight vector and hit it with the negative roots!

<u>Thm</u>. This gives you everything.

<u>What you get</u>:



Start with a highest weight vector

Hit it with $L_2 - L_1$ a bunch of times. How many until you get zero?

<u>Cheat</u>: Use what we've done before.

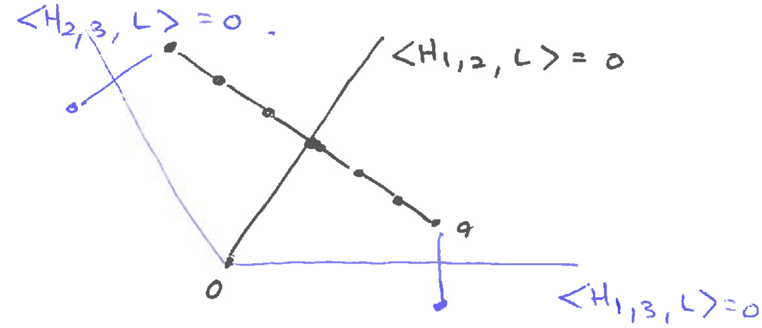
$$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$ is a subalgebra of $sl(3)$ iso. to $sl(2)$.

So use our $sl(2)$ - representation theory!
Let W be that line.
Eigenvalues of $H_{1,2}$ are integral and symmetric wrt zero.

So symmetric w.r.t. the line $\langle H_{1,2}, L \rangle = 0$ in the weight diagram!

(Also: this chain is at <u>right angles</u>, but this is <u>not</u> obvious yet, because we haven't specified how we embedded $\underline{h}^* \hookrightarrow \mathbb{R}^2$. We get "angles" once we have a nice bilinear form - here the <u>Killing form</u>.)

25.5 .



Step 1. Reflect in this line.

Step 2a. Do the same with $H_{1,3}$ and $H_{2,3}$.

Step 3. Play the same game all around the hexagon!

Step 4. Can also fill in the interior.

(Refer to the pictures in Fulton - Harris, Ch. 12.)

Proposition. Filling this in gives an irreducible rep'n; all the weight spaces have multiplicity 1.

Proposition. Any such weight diagram actually occurs as the weight diagram of an irrep of $sl(3)$.

So this is it! See Ch. 13 for some cool plethysm and geometry.

The Killing form. (Wilhelm Killing, 1847 - 1923)

This is a symmetric bilinear form $g \times g \to \mathbb{C}$

~~[illegible]~~

$$B(X, Y) = \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y) : g \to g.$$

Proposition. On $sl(n)$, $B(X, Y) = 2n \operatorname{tr}(XY)$.

Proof. First verify for $X = Y$.

Compute $\operatorname{Tr}((\operatorname{ad} X)^2)$.

Now
$$\begin{aligned}(\operatorname{ad} X)^2 Z &= (\operatorname{ad} X)(XZ - ZX) \\ &= XXZ - XZX - XZX + ZXX \\ &= X^2 Z + ZX^2 - 2XZX.\end{aligned}$$

25.6

If $X = (x_{ij})$, $(X^2)_{ij} = \sum_k x_{ik} x_{kj}$

$$(X^2 Z)_{ij} = \sum_\ell \left(\sum_k x_{ik} x_{k\ell} \right) z_{\ell j}.$$

The ij coeff of this is $\sum_k x_{ik} x_{ki}$.

$$\text{So } \operatorname{Tr}(X \to X^2 Z) = \sum_{ij} \sum_k x_{ik} x_{ki}$$

$$= n \sum_i \sum_k x_{ik} x_{ki} = n \operatorname{Tr}(X^2).$$

Similarly (check it!)

$\operatorname{Tr}(X \to Z X^2) = n \operatorname{Tr}(X^2)$ also

$\operatorname{Tr}(X \to X Z X) = (\operatorname{tr} X)^2 = 0$ because $X \in \underline{sl}(n)$.

So $B(X, X) = 2n \operatorname{tr}(X^2)$.

$$B(X, Y) = \frac{1}{2}\left(B(X + Y, X + Y) - B(X, X) - B(Y, Y) \right)$$

$$= n \left(\operatorname{tr}((X+Y)^2) - \operatorname{tr}(X^2) - \operatorname{tr}(Y^2) \right)$$

$$= 2n \operatorname{tr}(X Y).$$ (Remember this trick!)

An isomorphism between $\underline{h}^*$ and $\underline{h}$.

$$T_\varphi \in \underline{h} \longrightarrow \varphi \in \underline{h}^*$$

$$B(T_\varphi, H) = \varphi(H).$$

Given T_φ, this determines a functional.

Moreover, the association $T_\varphi \to \varphi$ is injective and hence surjective.

why? If $B(X, H) = 0$ for all $H \in \underline{h}$ and some $X \in H$ ...

For $sl(n)$, use above formula.

In general, $B(X, Y) = \sum_{\alpha \in \text{roots}} \alpha(X)\alpha(Y)$ and take $X = H$ above.

25.7

How to identify $\underline{h}^* \hookrightarrow \underline{h}$ here?

the functionals L_i are determined by

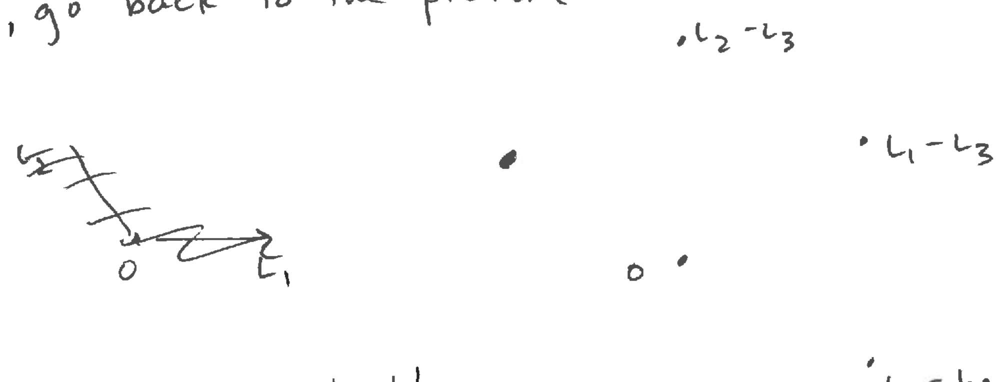
$$L_i(X) = \frac{1}{2n} \operatorname{Tr}\left(\begin{pmatrix} \text{1 in} \\ i\text{th} \\ \text{diagonal spot} \\ \text{0 everywhere else} \end{pmatrix} \cdot X \right)$$

So $L_1 - L_2 \hookrightarrow \frac{1}{6}\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}$

$L_1 - L_3 \hookrightarrow \frac{1}{6}\begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix}$

$L_2 - L_3 \hookrightarrow \frac{1}{6}\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}$

Now, go back to the picture.



Defines an inner product!

$$B(L_1 - L_2, L_2 - L_3) = B\left(\frac{1}{6}\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}, \frac{1}{6}\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}\right)$$

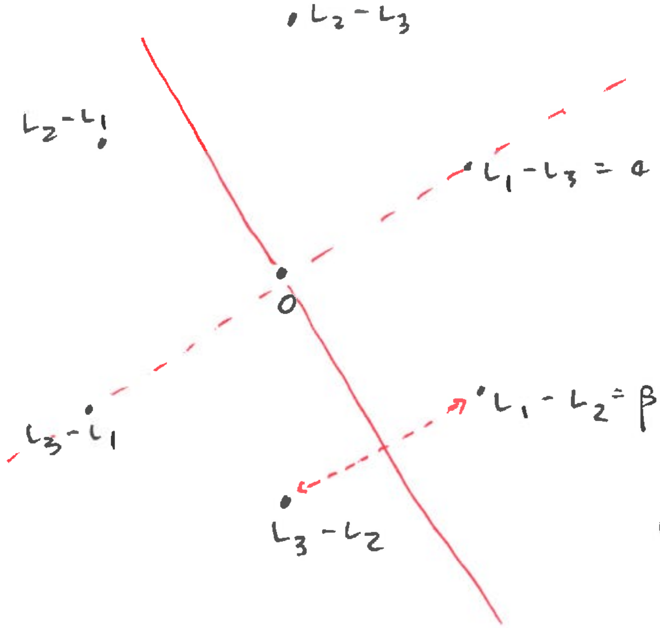
$$= \frac{1}{36} \cdot -1$$

$$B(L_1 - L_2, L_1 - L_2) = B\left(\frac{1}{6}\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}, \frac{1}{6}\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}\right)$$

$$= \frac{1}{36} \cdot 2.$$

25.8

We have a nice inner product now.



Each root α defines a perpendicular plane Ω_α

$$\Omega_\alpha = \beta \in \mathfrak{h}^* : B(\beta, \alpha) = 0.$$

Define W_α to be reflection in this plane:

$$W_\alpha(\beta) = \beta - \frac{2B(\alpha, \beta)}{B(\beta, \beta)} \alpha .$$

Then W_α induces a bijection on the roots.

Two fantastic conclusions.

(1) $\dfrac{2B(\alpha, \beta)}{B(\beta, \beta)}$ is an <u>integer</u>.

~~the angle between two roots is~~:

(2) Define the <u>Weyl group</u> W to be the group generated by all these automorphisms.

We see: these root systems have a <u>lot</u> of structure.