1.2.

The tangent space at the identity:

Let $\mathbb{R} \xrightarrow{\phi} SO(2,\mathbb{R})$

$$t \longrightarrow \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

We'll compute $\displaystyle \phi'(0) = \lim_{h\to 0} \frac{\begin{bmatrix} \cos h & -\sin h \\ \sin h & \cos h \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{h}$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

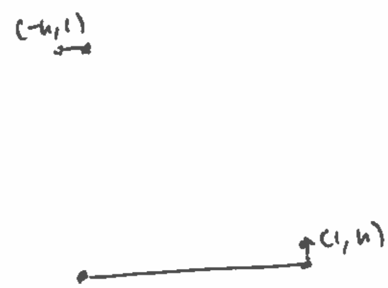
So the tangent space is $\left\{ \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} : t \in \mathbb{R} \right\}$.

(note: this is not a rigorous computation)

Suppose all we know about $SO_2(\mathbb{R})$ is that it contains I and this is its tangent space.

Then, very roughly, for small h

$$SO_2(\mathbb{R}) \ni \begin{bmatrix} 1 & -h \\ h & 1 \end{bmatrix}.$$



All powers of that should be in $SO_2(\mathbb{R})$ as well.
If n is big, then roughly $\begin{bmatrix} 1 & -t/n \\ t/n & 1 \end{bmatrix}^n$ should be in the group, for all $t \in \mathbb{R}$.

What is true is $\displaystyle SO_2(\mathbb{R}) \ni \lim_{n\to\infty} \begin{bmatrix} 1 & -t/n \\ t/n & 1 \end{bmatrix}^n$

1.3

$$= \lim_{n\to\infty}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix} + \frac{t}{n}\begin{bmatrix}0&-1\\1&0\end{bmatrix}\right)^n$$

$$= \begin{bmatrix}1&0\\0&1\end{bmatrix} + t\cdot\begin{bmatrix}0&-1\\1&0\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}0&-1\\1&0\end{bmatrix}^2 + \frac{t^3}{3!}\begin{bmatrix}0&-1\\1&0\end{bmatrix}^3 + \cdots$$

$$=: \exp\left(\begin{bmatrix}0&-t\\t&0\end{bmatrix}\right).$$

Now, what is that?

If we use the isomorphism

$$\left\{\begin{pmatrix}a&-b\\b&a\end{pmatrix} : a, b \in \mathbb{R}\right\} \longrightarrow \mathbb{C}$$

$$\begin{pmatrix}a&-b\\b&a\end{pmatrix} \longrightarrow a+bi$$

(exercise: check the multiplication

we have

$$\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix} \longrightarrow \cos\theta + i\sin\theta$$

$$\begin{pmatrix}0&-t\\t&0\end{pmatrix} \longrightarrow it$$

and this says that ~~every element of $SO(2,\mathbb{R})$ can~~ ~~be~~ $e(it) \in \{\cos\theta + i\sin\theta : \theta\in\mathbb{R}\}$.

Indeed, they're equal and

$SO_2(\mathbb{R}) = \exp(T)$ where T is the tangent space.

1.4.

<u>Definition</u>. If K is any field, (this class: use $\mathbb{R}, \mathbb{C}$)

$GL_n(K)$ = {all invertible $n \times n$ matrices w/ entries in K}.
or $GL(n,K)$

$SL_n(K)$ = {same, w/ <u>determinant</u> 1}

$M_n(K)$ = <u>all</u> $n \times n$ matrices w/ entries in K.

<u>Definition</u>. A matrix Lie group is a subgroup $G \leq GL_n(\mathbb{C})$ which is closed under the subspace topology. (wrt $GL_n(\mathbb{C})$)

What the latter statement means.

We have $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ as vector spaces

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \longrightarrow (a_{11}, a_{12}, \cdots, a_{nn}).$$

The right side has the usual topology and we give the same to $M_n(\mathbb{C})$.

So, if $\{A_1, A_2, A_3, \dots\}$ is a seq. of matrices in $M_n(\mathbb{C})$, then the sequence <u>converges</u> <u>to a matrix</u> A if and only if each entry ~~conv~~ converges to the corresponding entry of A.

The closure condition means, if $A_1, A_2, A_3, \dots \in G$ and $A_i \to A$ with <u>$A \in GL_n(\mathbb{C})$</u>, then $A \in G$.

(Note: if $A_i \longrightarrow A$ but $A \notin GL_n(\mathbb{C})$, no assumption.)

1.5. See p. 4 for some non-examples.

Example. The unitary group $U_n(\mathbb{C})$ is

(1) $\{A \in M_n(\mathbb{C}) : \sum_{l=1}^{n} \overline{A_{lj}} A_{lk} = \delta_{jk}$ for all $j,k\}$.
(columns are orthonormal)

Here $\bar{\cdot}$ is the complex conjugate.

For any matrix $A \in M_n(\mathbb{C})$, its adjoint A^* is its conjugate transpose

and we can rewrite (1) as

(2) $\{A \in M_n(\mathbb{C}) : \sum_{l=1}^{n} (A^*)_{jl} A_{lk} = \delta_{jk}$ for all $j,k\}$

(3) $= \{A \in M_n(\mathbb{C}) : A^* A = I\}$.

Claims. (1) Unitary matrices are invertible
(2) A is unitary iff $A^* = A^{-1}$
] (immediate from (3))
(3) $U_n(\mathbb{C})$ is actually a group.

Proofs. ~~If $A^* A = I$,~~

Note that $(AB)^* = B^* A^*$ because $(AB)^T = B^T A^T$

$$\overline{AB} = \bar{A}\,\bar{B}$$

$$(\bar{A})^T = (\overline{A^T}).$$

If $A, B \in U_n(\mathbb{C})$, then

$(AB)^*(AB) = B^* A^* A B = B^* B = I$ so $AB \in U_n(\mathbb{C})$.

Similarly $(A^{-1})^* A^{-1} = (A^{-1})^* A^* = (A^* A^{-1})^* = I^* = I$.

So $A^{-1} \in U_n(\mathbb{C})$.

1.6
=2.1

Def. $SU_n(\mathbb{C}) = \{A \in U_n(\mathbb{C}) : \det A = 1\}$
the special unitary group
$= SL_n(\mathbb{C}) \cap U_n(\mathbb{C})$.

Proposition.
(1) If A_i is a seq of matrices in $U_n(\mathbb{C})$ and $A_i \to A$ for some $A \in GL_n(\mathbb{C})$, then $A \in U_n(\mathbb{C})$.
(i.e: ~~A is~~ $U_n(\mathbb{C})$ is closed in $GL_n(\mathbb{C})$
i.e: $U_n(\mathbb{C})$ is a matrix Lie group)

(2) Same for $SU_n(\mathbb{C})$.

Exercise.
(1), (2). See above,
(3) If $A_i \to A$ for some $A_i \in SU_n(\mathbb{C})$ and $A \in M_n(\mathbb{C})$ then $A \in SU_n(\mathbb{C})$. So $SU_n(\mathbb{C})$ is a closed subgroup of $M_n(\mathbb{C})$.
(4) The analogue of (3) is not true for $U_n(\mathbb{C})$!

Unitary groups and inner products.

Let $\langle -, - \rangle$ denote the standard inner product on $\mathbb{C}^n$ (or $\mathbb{R}^n$):

$$\langle x, y \rangle = \sum_j \overline{x_j}\, y_j .$$

Proposition. $\langle x, Ay \rangle = \langle A^* x, y \rangle$.

Proof. Exercise (Compute both sides; they're the same ugly thing)

So: $\langle Ax, Ay \rangle = \langle A^* A x, y \rangle$ so
A is unitary $\Rightarrow \langle Ax, Ay \rangle = \langle x, y \rangle$ (A preserves inner products)

1.7 = 2.2

Claim. If $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{C}^n$ then A is unitary.

Proof. We have $\langle Ax, Ay\rangle = \langle A^*Ax, y\rangle$ (even if A is not unitary)

so we are assuming $\langle A^*Ax, y\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{C}^n$

i.e. $\langle (A^*A - I)x, y\rangle = 0$, for all $x, y \in \mathbb{C}^n$.

By nondegeneracy of the inner product $(A^*A - I)x = 0$ for all $x \in \mathbb{C}^n$.

So $A^*A - I = 0$. Done

Deteminants: For any A, $\det(A^*) = \det(\overline{A^T})$

$$= \overline{\det(A^T)}$$

$$= \overline{\det(A)}.$$

So, if A is unitary,

$\det(A^*A) = |\det A|^2 = \det I = 1$.

So $|\det A| = 1$.

Exercise.

$$SU_2(\mathbb{C}) = \left\{ \begin{pmatrix} a+di & -b-ci \\ b-ci & a-di \end{pmatrix} . \ a, b, c, d \in \mathbb{R} \right\}.$$

1.8 ⇒ 2.3

Orthogonal groups. Same thing with $\mathbb{R}$.
So no conjugate transpose.
Just the transpose.

The orthogonal group $O_n(\mathbb{R})$ is the group of $n \times n$ matrices A satisfying the following equivalent conditions.

(1) $A^T A = I$

(2) $A^T = A^{-1}$

(3) A preserves the inner product on $\mathbb{R}^n$, i.e.
$\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^n$.
In particular A ~~sends~~ preserves angles
sends ONB's to ONB's.

(4; not proved for now) is a combination of rotations and reflections).

~~The~~ Note, if $A \in O_n(\mathbb{R})$, $\det(A^T A) = (\det A)^2 = 1$
So $\det A = \pm 1$.

The special orthogonal group $SO_n(\mathbb{R}) := O_n(\mathbb{R}) \cap SL_n(\mathbb{R})$.
We have an exact sequence

$$1 \longrightarrow SO_n(\mathbb{R}) \longrightarrow O_n(\mathbb{R}) \xrightarrow{\det} \pm 1 \longrightarrow 1.$$

We have analogously the orthogonal groups $O_n(\mathbb{C})$ satisfying $A^T A = I$ and $\langle Ax, Ay \rangle = \langle x, y \rangle$
with now $(x, y) = \sum_i x_i y_i$.

Works for any field too.

2.4.

An alternative proof of one of the claims. (Stillwell p. 50)

Proposition. If $A \in M_n(\mathbb{R})$ then

$$A^T A = 1 \longleftrightarrow \langle Ax, Ay\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}^n.$$

Proof. $A^T A = 1 \longleftrightarrow$ (row i of A^T) $\cdot$ (col j of A) $= \delta_{ij}$

$\longleftrightarrow$ (col i of A) $\cdot$ (col j of A) $= \delta_{ij}$

$\longleftrightarrow$ cols of A are an orthonormal basis

$\longleftrightarrow$ A-images of the standard basis form an ONB

$\longleftrightarrow$ A preserves the inner product

because $\langle Ae_i, Ae_j\rangle = \langle e_i, e_j\rangle$

so $\langle Ax, Ay\rangle = \langle x, y\rangle$ is true for a set of x, y forming a basis for $\mathbb{R}^n$.

So orthogonal matrices are isometries:

They preserve distances.

Here $d(x,y) = |x-y| = \langle x-y, x-y\rangle$

so $d(x,y) = d(Ax, Ay)$

2.5 Conversely any isometry is orthogonal, i.e.

$$\langle Ax, Ax\rangle = \langle x, x\rangle \ \forall x \Rightarrow \langle Ax, Ay\rangle = \langle x, y\rangle \ \forall x, y.$$

Why is this? Write out

$$\langle A(x+y), A(x+y)\rangle = \langle x+y, x+y\rangle$$

and FOIL.

$$\langle Ax, Ax\rangle + 2\langle Ax, Ay\rangle + A\langle y, y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle$$

//

Generalized orthogonal groups (skipped for now).

(3. Review)

Symplectic Groups:

Define a bilinear form ω on $\mathbb{R}^{2n}$:

$$\omega(x, y) = \sum_{j=1}^{n} (x_j y_{n+j} - x_{n+j} y_j).$$

This is skew-symmetric or alternating: $\omega(x, y) = -\omega(y, x)$.

In particular, $\omega(x, x) = 0$.

Def. The (real) ~~symmetric~~ symplectic group $Sp_n(\mathbb{R})$ is

$$\{ A \in GL_{2n}(\mathbb{R}) : \omega(Ax, Ay) = x, y \text{ for all } x, y \in \mathbb{R}^{2n} \}.$$

Note that $Sp_n(\mathbb{R})$ consists of $2n \times 2n$ matrices.

This is annoying. Indeed, some write $Sp_{2n}(\mathbb{R})$.

Certainly $2n$ has to be even.

2.6. Writing

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

we have $\omega(x,y) = \langle x, \Omega y\rangle$.

So, $\omega(Ax, Ay) = \omega(x,y)$

$\longleftrightarrow \langle Ax, \Omega A y\rangle = \langle x, \Omega y\rangle$.

Exercise. This is equivalent to $-\Omega A^T \Omega = A^{-1}$.
(NO UGLY COMPUTATIONS. Use $\langle x, Ay\rangle = \langle A^* x, y\rangle$
which in this case just says
$\langle x, Ay\rangle = \langle A^T x, y\rangle$.)

Take determinant above: $\det A = \pm 1$, for $A \in Sp_n(\mathbb{R})$.
In fact $\det A = 1$. Not obvious!

There is also the complex symplectic group
$Sp_n(\mathbb{C})$ $2n \times 2n$ matrices preserving the same form or equivalently with $-\Omega A^T \Omega = A^{-1}$.
(no A^* here.)

And finally the compact symplectic group
$$Sp(n) = Sp(n;\mathbb{C}) \cap U(2n).$$
This notation is appalling.

More on these groups later.

(skipped for now)

2.7 The Euclidean group $E(n)$.

All transformations of $\mathbb{R}^n$ that are the composition of a translation and an orthogonal linear transformation.

i.e. $E(n) = \{$transformations $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ of the form $y \longrightarrow Ry + x$ for some $x \in \mathbb{R}^n$, $R \in O(n)\}$.

Note $E(n) \not\subseteq GL_n(\mathbb{R})$. These are not linear trans.

But you can cheat. $E(n)$ is isomorphic to the closed subgroup of $GL_{n+1}(\mathbb{R})$ given by

$$\begin{pmatrix} \boxed{O_n} & \begin{matrix} * \\ * \\ \vdots \\ * \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix}.$$

(Exercise: prove this).

The Heisenberg group.

All 3×3 matrices $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$.

3.1

More on $Sp(n)$.

Define a conjugate-linear map $J : \mathbb{C}^{2n} \longrightarrow \mathbb{C}^{2n}$

$$(\alpha, \beta) \to (-\bar{\beta}, \bar{\alpha})$$

with $\alpha \in \mathbb{C}^n$, $\beta \in \mathbb{C}^n$.

Here conjugate linear means $J(v + w) = J(v) + J(w)$

$$J(\lambda v) = \bar{\lambda} J(v).$$

Then we have, for all $z, w \in \mathbb{C}^{2n}$ that

$$\omega(z, w) = \langle Jz, w \rangle.$$

The LHS is $\sum_{j=1}^{n} (z_j w_{n+j} - z_{n+j} w_j)$

The RHS is $\langle (-\overline{z_{n+1}}, \dots, -\overline{z_{2n}}, \overline{z_1}, \dots, \overline{z_n}),$

$$(w_1, \dots, w_n, w_{n+1}, \dots, w_{2n}) \rangle$$

$$= -z_{n+1} w_1 - \dots - z_{2n} w_n + z_1 w_{n+1} + \dots + z_n w_{2n}.$$

~~Now $\langle x, Ay \rangle = \langle A^* x, y \rangle$~~ (def of adjoints)

~~$\langle Ax, Ay \rangle = \langle A^* A x, y \rangle$~~ (true but not necessary here)

~~$= \langle x, y \rangle$ (if A is unitary;~~

~~recall $Sp(n) = U(2n) \cap Sp(n; \mathbb{C})$~~)

So $\langle Jz, w \rangle = \omega(z, w) = -\omega(w, z)$

$$= -\langle Jw, z \rangle$$

$$= -\overline{\langle z, Jw \rangle}$$

and also $J^2 = -I$.

3.2.

Proposition. If U belongs to $U(2n)$ then

$$U \in Sp(n) \longrightarrow U \text{ commutes with } J.$$

(i.e. $U \in Sp(n, \mathbb{C})$)

Proof. For $U \in U(2n)$, we have for all $z, w \in \mathbb{C}^{2n}$,

$$\omega(Uz, Uw) = \langle JUz, Uw\rangle = \langle U^* J U z, w\rangle = \langle U^{-1} J U z, w\rangle$$

and
$$\omega(z, w) = \langle Jz, w\rangle$$

So that these are the same iff $U^{-1}JU = J$, i.e. $JU = UJ$.

So then for $U \in Sp(n)$, U commutes with J.
U also commutes with ~~multiplication by~~ i
the scalar matrix

$$i = \begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix}$$

and iJ.

Note that $J(iz) = -iJ(z)$ (J is conjugate linear).

Write $\vec{i} = i$, $\vec{j} = J$, $\vec{k} = iJ$.

Then
$$i^2 = -I$$
$$J^2 = -I$$
$$(iJ)^2 = iJiJ = -i \cdot i \cdot J \cdot J = -I$$
$$iJ = -Ji$$
$$i(iJ) = -(iJ)i$$
$$J(iJ) = -(iJ)J.$$

So we've checked that $\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1$,

$$\vec{i}\vec{j} = -\vec{j}\vec{i} = \vec{k}$$
$$\vec{i}\vec{k} = -\vec{k}\vec{i}, \quad \vec{k}\vec{i} = -\vec{i}\vec{k}.$$

3.3 So what?

Recall that the Hamiltonian quaternions are

$\mathbb{H} = \{$ ~~vector space~~ algebra gen. by $1, \vec{i}, \vec{j}, \vec{k}$ subject to above relations$\}$

and so we have made $\mathbb{C}^{2n}$ into a "vector space" over $\mathbb{H}$.

In other words, we write

$$\vec{i} \cdot z = iz$$

$$\vec{j} \cdot z = Jz$$

$$\vec{k} \cdot z = iJz$$

and extend by linearity, and $U \in Sp(n)$ commutes with $\vec{i}, \vec{j}, \vec{k}$, hence any $a + b\vec{i} + c\vec{j} + d\vec{k}$.

Therefore U is "quaternion linear": $U(\gamma z) = \gamma U(z)$ for any quaternion γ.

All this proves:

Proposition. A $2n \times 2n$ matrix U belongs to $Sp(n)$ iff U is quaternion linear and preserves the norm. (i.e. the inner product)

3.4. Another perspective on the same thing. (Stillwell)

Recall that complex numbers can be represented by 2×2 real matrices.

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then $i^2 = -1$ so we're golden.

We can also represent quaternions as 2×2 complex matrices:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \vec{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad \vec{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

i.e. $$a + b\vec{i} + c\vec{j} + d\vec{k} = \begin{pmatrix} a+di & -b-ci \\ b-ci & a-di \end{pmatrix}$$

Check, $\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1$; $\vec{i}\vec{j} = -\vec{j}\vec{i} = \vec{k}$

$\vec{j}\vec{k} = -\vec{k}\vec{j} = \vec{i}$

$\vec{k}\vec{i} = -\vec{i}\vec{k} = \vec{j}$.

Properties. (1) If q is a quaternion,

$$\det(q) = (a+di)(a-di) + (b-ci)(b+ci)$$
$$= a^2 + b^2 + c^2 + d^2$$

and so $|q_1 q_2| = |q_1||q_2|$ by multiplicativity of determinants.

(2) $$q^{-1} = \frac{a - b\vec{i} - c\vec{j} - d\vec{k}}{|q|}$$ (Invert the matrix!)

(3) The unit quaternions $\{q : |q| = 1\}$ form the 3-sphere $S^3 \subseteq \mathbb{R}^4$.

3.5. **Proposition.** Let n, m be integers which are sums of four integer squares. Then so is $n \cdot m$.

Proof. We can write $n = |q_1|$ for a quaternion q_1 with integer coeffs

$m = |q_2|$ "

and so $nm = |q_1 q_2|$. Done.

This is true for two also, but not three.

Theorem (Legendre) Every positive integer is the sum of four squares.

Sketch of proof: (1) By above, reduce to every prime p.

(2) Prove that mp = sum of four squares for some $m \geq 1$.

(3) Infinite descent.

There is also an inner product for quaternions. Given $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n) \in \mathbb{H}^n$,

$\langle (p_1 \cdots p_n), (q_1 \cdots q_n) \rangle = \bar{p}_1 q_1$ (Hall) (ARRRRGGHHH!!

or $p_1 \bar{q}_1$ (Stillwell)

Def. $Sp(n) := \{ A \in M_n(\mathbb{H}) : \langle Av, Aw \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{H}^n \}$.

3.6. How to reconcile these?

Restriction of scalars.

We represented complex numbers as 2×2 real matrices.

Prop. There exists an injective homomorphism

$$GL_n(\mathbb{C}) \xrightarrow{Res_{\mathbb{C}/\mathbb{R}}} GL_{2n}(\mathbb{R}) \quad \text{for any } n \geq 1.$$

Proof. $\mathbb{C}^n$ is certainly a $2n$-dimensional real vector space, and if ϕ is $\mathbb{C}$-linear it is certainly $\mathbb{R}$-linear as well.

We had an injection

$$\mathbb{C}^\times = GL_1(\mathbb{C}) \longrightarrow GL_2(\mathbb{R})$$

$$a + bi \longrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\cos\theta + i\sin\theta \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and just duplicate this. e.g.

$$\begin{pmatrix} a+bi & c+di \\ e+fi & h+ji \end{pmatrix} \longrightarrow \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ e & f & h & j \\ -f & e & -j & h \end{pmatrix}$$

We can say still more. A linear transformation $\phi \in GL_n(\mathbb{C})$ is (as required) i-linear if $\phi(iv) = i\phi(v)$ for all v.

So we have to have

$$Res_{\mathbb{C}/\mathbb{R}}\phi\left(Res_{\mathbb{C}/\mathbb{R}} \begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix} (v) \right)$$

$$= Res_{\mathbb{C}/\mathbb{R}} \begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix} \left(Res_{\mathbb{C}/\mathbb{R}} \phi(v) \right) \quad \forall v,$$

so we have to demand that

~~Res~~ $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\phi$ commutes with $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\begin{pmatrix} i & & & \\ & i & & \\ & & \ddots & \\ & & & i \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

Call this J_n

Proposition. This property characterizes $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$, i.e.

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, GL_n(\mathbb{C}) = \{A \in GL_{2n}(\mathbb{R}) : AJ_n = J_nA\}.$$

The converse is proved the same way.
Given $A \in GL_{2n}(\mathbb{R})$ with $AJ_n = J_nA$, inverting $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ yields a function $\mathbb{C}^n \to \mathbb{C}^n$ which we check is a $\mathbb{C}$-linear transformation (because it commutes with i).