Homework 9, Math 702 – Frank Thorne (thorne@math.sc.edu)

You are welcome and encouraged to collaborate, but please write up your own solutions.
Due Friday, March 2, 2018.

1. Prove the following theorem regarding the existence of a **Smith Normal Form**:

   Theorem. Let A be a nonzero $m \times n$ matrix over a PID R. Then, there exist invertible $m \times m$ and $n \times n$ matrices S and T, so that the matrix $B := SAT$ satisfies the following properties:

   - B is a diagonal matrix, in the sense that $B_{ij} = 0$ except when $i = j$ and $1 \leq i, j \leq r$ for some r.
   - We have $B_{ii} \mid B_{ii+1}$ for all $1 \leq i < r$.

2. If $A \in \mathrm{GL}_3(\mathbb{Q})$ and $A^8 = I$, prove that in fact $A^4 = I$.

3. - Determine, up to conjugacy, all matrices $A \in \mathrm{GL}(3, \mathbb{F}_3)$ with $A^3 = I$.
   - Determine, up to conjugacy, all matrices $A \in \mathrm{GL}(3, \overline{\mathbb{F}_3})$ with $A^3 = I$. (There are not necessarily more, because the conjugacy is defined within a different group.)

4. (Don't hand in) The polynomial $x^3 - 2x - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion. Let θ be a root.

   Do a bunch of arithmetic in the field $\mathbb{Q}(\theta)$, until you feel like you know what you're doing. For example, write θ^3, θ^4, θ^{-1}, $\frac{1}{1+\theta}$, etc. in the form $a + b\theta + c\theta^2$. Repeat with other polynomials until you get bored.

5. Let $\mathbb{F}_9 := \mathbb{F}_3[x]/(x^2 + x - 1)$ be the *field with nine elements.* (It is unique up to isomorphism, but you don't need to prove this.)

   Then $\mathbb{F}_9^\times$ is a group with 8 elements. (The group consists of all nonzero elements with multiplication as the operation.) Determine the structure of this group. It must be isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/4 \times \mathbb{Z}/2$, or $\mathbb{Z}/8$ – determine which.

6. Determine the degrees of $\mathbb{Q}(\sqrt{5} + \sqrt[3]{7})$ and $\mathbb{Q}(\sqrt{3 + 2\sqrt{2}})$ over $\mathbb{Q}$.

7. Determine the splitting field and its degree over $\mathbb{Q}$ for $x^6 - 4$.