Homework 3, Math 701 – Frank Thorne (thorne@math.sc.edu)

Instructions: You are welcome and encouraged to collaborate, but please write up your own solutions.
Due Friday, September 29, 2017.

1. (a) Prove (see 9.2 of the lecture notes for definition) that $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half plane $\mathbb{H}$ by linear fractional transformations.
   (b) Characterize, as accurately as you can, the size of the stabilizers of this action. (At a minimum, give three example points $z \in \mathbb{H}$ whose stabilizers have different sizes.)
2. Compute, with proof, all subgroups of $\mathrm{Sym}(3)$, and determine which of them are normal.
3. Let G be the group of rigid motions of a cube.
   (a) The position of a cube is determined by where each of its faces are. (You may accept this as 'geometrically obvious'). Explain why this yields an injective homomorphism $G \to \mathrm{Sym}(6)$.
   (b) G also acts on the three-element set of *pairs* of opposite faces. Prove that the resulting homomorphism $G \to \mathrm{Sym}(3)$ is *not* injective.
   (c) By describing yet another action of G on (... something associated to the cube ...!), prove that the group of rigid motions of a cube is isomorphic to $\mathrm{Sym}(4)$.
   (d) Combining the last two problems yields a surjective homomorphism $\mathrm{Sym}(4) \to \mathrm{Sym}(3)$. Describe it explicitly and compute its kernel.
4. Exhibit a group G and a subset $A \subset G$ for which $C_G(A) \neq N_G(A)$.
5. (a) (Bonus!) Do the exercise on p. 10.3 of the lecture notes concerning the action of $\mathrm{GL}_2(\mathbb{C})$ on the space V of *binary cubic forms*.
   (b) (Double Bonus!) Compute all of the orbits of this action (hint: there are four), and determine the stabilizer of each (which will be defined only up to conjugacy).
   (c) (Triple Bonus!) Find a polynomial P of the variables a, b, c, d so that the largest orbit consists precisely of those $v \in V$ with $P(v) \neq 0$.