Homework 1, Math 701 – Frank Thorne (thorne@math.sc.edu)

Instructions: You are welcome and encouraged to collaborate, but please write up your own solutions.

Due Friday, September 15, 2017.

1. Describe a couple of 'interesting' examples of vector spaces, and homomorphisms to or from your vector spaces.

2. Give a proof that every basis of $\mathbb{R}^2$ has exactly two vectors. **Your proof should be as minimal as possible**; in particular, don't refer to any propositions that say anything about 'dimension'.

3. (**Row Rank Equals Column Rank**) The **row rank** and **column rank** of an $n \times m$ matrix A are the dimensions of the subspaces of F^n and F^m, respectively, spanned by its rows and columns respectively.

   Prove[1] that the row rank and column rank are equal as follows:

   - Show that the column rank is the smallest number r for which the homomorphism ϕ corresponding to A can be written as a composition of two linear maps $F^m \to F^r \to F^n$.
   - Conclude that r is the smallest number for which one can write $A = BC$, with B an $n \times r$ and C an $r \times m$ matrix respectively.
   - Taking transposes, conclude the result.

4. Let V be a vector space, possibly infinite dimensional.

   (a) Prove that the map
   $$v \to \{f \to f(v)\}$$
   defines an injective map from V to the double dual $V^{**} = (V^*)^*$.

   (b) Prove that this map is not surjective if V is infinite dimensional.

5. Let V be a finite dimensional vector space, and let A and B be the matrices of a linear transformation ϕ with respect to two different bases.

   Prove that $B = MAM^{-1}$ for some matrix M, and describe M explicitly.

6. Exhibit a 2×2 matrix with real entries which does not have any eigenvalues over $\mathbb{R}$. (Better still: can you characterize all such matrices?)

[1]Credit: I learned of this proof from Marc van Leeuwen