Homework 7 - Math 580, Frank Thorne (thornef@mailbox.sc.edu)

Due by the final exam

(1) Do Exercises 1, 2, 4, 5, and 6 in Chapter 16. (These are the exercises in the middle of the chapter, not the problems at the end of the section.)

(2) Dudley, p. 133, 1.

(3) Dudley, p. 148, 1, 3, 5, 6.

Bonus problems. (Not as hard as they look!)

(1) In this problem you will find **all** integer solutions to the equation $x^2 + 2y^2 = z^2$. Your proof will be **stunningly beautiful** and will demonstrate that **an ellipse is birational to a line.**

(a) Explain why it is enough to find all *rational* solutions to $x^2 + 2y^2 = 1$.

(b) Graph the ellipse $x^2 + 2y^2 = 1$ and the line $x = 1$.

(c) Explain why any line through the point $(-1, 0)$, other than the vertical line, intersects the ellipse $x^2 + 2y^2 = 1$ and the line $x = 1$ each at exactly one point.

(d) Suppose that (a, b) is a point on $x^2 + 2y^2 = 1$ with a and b rational. Draw (a, b) on your graph. Graph the line between $(-1, 0)$ and (a, b) and find its equation. Find the intersection point $(1, c)$ with $c = 1$, and verify that it is rational.

(e) Suppose that $(1, c)$ is a point on $x = 1$ with c rational. Draw $(1, c)$ on your graph. Graph the line between $(-1, 0)$ and $(1, c)$ and find its equation. Find the intersection point (a, b) with $x^2 + 2y^2 = 1$, and verify that both a and b are rational.

(f) Explain why this gives a correspondence between rational solutions (a, b) to $x^2 + 2y^2 = 1$ and rational numbers c.

(g) Plugging in a few small values of c, find a few integral solutions to the equation $x^2+2y^2 = z^2$.

(2) Now, do the same for the equation $x^2 - y^2 = z^2$. Abbreviated instructions:

(a) Explain why it is enough to find rational points on the hyperbola $x^2 - y^2 = 1$. Graph this hyperbola.

(b) In this problem, the line $x = 0$ takes the place of $x = 1$, and the point $(1, 0)$ takes the place of $(-1, 0)$. As before, find a correspondence between rational solutions (a, b) to $x^2 - y^2 = 1$ and points $(0, -c)$ on the line. This time, to make your life easier, restrict your attention to positive a, b, c with $c > 1$.

(c) As before, find a few solutions to $x^2 - y^2 = z^2$. Although our work feels new here, it is entirely redundant with something we did before! Explain why.