Please work without books, notes, calculators, or any assistance from others.

1. (13 points) If \( a \) is any odd integer and \( b \) is any even integer, prove that \( 2a + 3b \) is even. (For this problem, use only the definitions of even and odd, and do not appeal to any previously established properties of even and odd integers.)

Answer 1: You know that \( a \) is odd, and therefore \( a = 2r + 1 \) for some integer \( r \). You know that \( b \) is even, and therefore \( b = 2s \) for some integer \( s \). Therefore,

\[
2a + 3b = 2(2r + 1) + 3(2s) = 4r + 2 + 6s = 2(2r + 3s + 1).
\]

We know that \( 2r + 3s + 1 \) is an integer, so that \( 2a + 3b \) is twice an integer, and therefore is even.

Answer 2: (This incorporates a small shortcut that you may have noticed.) You know that \( b \) is even, and therefore \( b = 2s \) for some integer \( s \). Therefore,

\[
2a + 3b = 2a + 3(2s) = 2(a + 3s).
\]

We know that \( a + 3s \) is an integer, so that \( 2a + 3b \) is twice an integer, and therefore is even.

2. (13 points) Suppose that the product of three positive real numbers \( x, y, \) and \( z \) is at least 70. Prove that at least one of \( x, y, \) and \( z \) is greater than 4.

We argue by contradiction. Suppose that \( x, y, \) and \( z \) are all positive integers which are less than or equal to 4. Then,

\[
x \cdot y \cdot z \leq 4 \cdot 4 \cdot 4 = 64,
\]

so that \( xyz < 64 \). However, this contradicts the assumption that \( xyz \geq 70 \). Therefore, at least one of \( x, y, \) and \( z \) is greater than 4.

3. (13 points) Determine whether the following statement is true or false, and prove or disprove it: If an integer \( a \) is of the form \( 5n + 1 \) for some integer \( n \), then \( a^2 \) is of the form \( 25m + 1 \) for some integer \( m \).

False. We exhibit a counterexample. Let \( n = 1 \) so that \( a = 6 \). Then, \( a^2 = 36 = 25 + 11 \). By the unique division-with-remainder theorem, \( a^2 \) cannot be of the form \( 25m + 1 \) if it is of the form \( 25b + 11 \) (where \( b = 1 \)).

4. (14 points) Prove that \( \sqrt[3]{4} \) is irrational.

You may use the following statement without proving it: For all integers \( a \), if \( a^3 \) is even then \( a \) is even.

Proof: Suppose to the contrary that \( \sqrt[3]{4} \) is rational, so that we can write it as a fraction \( \frac{a}{b} \), written where \( a \) and \( b \) are both positive and have no common factor. Then, cubing both sides
of \( \sqrt[3]{4} = \frac{a}{b} \), we get \( 4 = \frac{a^3}{b^3} \), so that \( 4b^3 = a^3 \). Thus, \( a^3 \) is even, and so \( a \) is also even, and we can write \( a = 2r \) for some integer \( r \). We have \( 4b^3 = (2r)^3 \), so that \( b^3 = 2r^3 \). Therefore, \( b^3 \) is even, and hence \( b \) is even also.

But this shows that \( a \) and \( b \) are both even and have the common factor 2, contrary to assumption. This is a contradiction; therefore, \( \sqrt[3]{4} \) is irrational.

5. (14 points) Prove that \( \lim_{x \to 3} (2x + 1) = 7 \).

Proof: Suppose that \( \epsilon > 0 \) is given.

[Aside: Not needed for proof, but shows you how to pick \( \delta \). If \( 2x + 1 = 7 + \epsilon \), then \( x = 3 + \epsilon/2 \), and similarly if \( 2x + 1 = 7 - \epsilon \), then \( x = 3 - \epsilon/2 \). So we should pick \( \delta = \epsilon/2 \), or anything smaller.]

Choose \( \delta = \epsilon/2 \). Suppose that we are given \( x \) with \( |x - 3| < \delta \), i.e., \( 3 - \epsilon/2 < x < 3 + \epsilon/2 \). Then, we have \( 2(3 - \epsilon/2) + 1 < 2x + 1 < 2(3 + \epsilon/2) + 1 \), i.e., \( 7 - \epsilon < 2x + 1 < 7 + \epsilon \). In other words \( |(2x + 1) - 7| < \epsilon \) whenever \( |x - 3| < \delta \). By definition, \( \lim_{x \to 3} (2x + 1) = 7 \) as desired.

6. (14 points) Prove, for all integers \( n \geq 1 \), that

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n + 1)} = \frac{n}{n + 1}.
\]

We prove this by induction. Let \( P(n) \) be the claim that

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n + 1)} = \frac{n}{n + 1}.
\]

Then \( P(1) \) is true because both sides are equal to \( 1/2 \). Suppose now that \( P(n) \) is true for some particular \( n \). We need to show that \( P(n + 1) \) is true. The left hand side of \( P(n + 1) \) is

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n + 1)} + \frac{1}{(n + 1) \cdot (n + 2)}.
\]

By our inductive hypothesis (that \( P(n) \) is true), this is equal to

\[
\frac{n}{n + 1} + \frac{1}{(n + 1)(n + 2)} = \frac{n(n + 2)}{(n + 1)(n + 2)} + \frac{1}{(n + 1)(n + 2)}.
\]

This is equal to

\[
\frac{n(n + 2) + 1}{(n + 1)(n + 2)} = \frac{n^2 + 2n + 1}{(n + 1)(n + 2)} = \frac{(n + 1)^2}{(n + 1)(n + 2)} = \frac{n + 1}{n + 2},
\]

which is the right hand side of \( P(n + 1) \). Therefore \( P(n + 1) \) is true, and hence \( P(n) \) is true for all \( n \geq 1 \) by induction.

7. (14 points) Prove that \( 1 + 3n \leq 4^n \) for every integer \( n \geq 0 \).

We argue by induction. Let \( P(n) \) be the claim \( 1 + 3n \leq 4^n \). Then \( P(0) \) is true because both sides are equal to 1. Suppose now that \( P(n) \) is true for some particular \( n \). We want to prove that \( P(n + 1) \) is true.

The left side of \( P(n + 1) \) is equal to \( 1 + 3(n + 1) = (1 + 3n) + 3 \). By induction, this is less than \( 4^n + 3 \leq 4^n + 3 \cdot 4^n = 4^{n+1} \), so that \( P(n + 1) \) is true. The result follows by induction.
8. (5 points) Let $S$ be the set of integers divisible by 3, and let $T$ be the set of integers divisible by 6. Do we have $S \subseteq T$? Do we have $T \subseteq S$?

[For this problem, you do not have to give a proof or explanation (you should know how to – but time is short), but if your answer is wrong, this might be worth partial credit.]

We have $T \subseteq S$ but not $S \subseteq T$. If $x$ is an integer divisible by 6, then $x = 6r$ for some integer $r$, so that $x = 3(2r)$, so that $x$ is a multiple of 3 (i.e., an element of $S$). To see that $S \nsubseteq T$, observe that 3 is in $S$ but not $T$. 