Midterm Examination 2 - Math 547, Frank Thorne (thorne@math.sc.edu)

Due Monday, April 20 at the beginning of class

Instructions:

- This is a **timed, take-home, closed-book exam.**
- The exam covers Chapters 19, 20, and 22 of Saracino. It also implicitly covers Chapters 16-18 because the material of the later chapters builds on this.
- You may take the exam at any time and place of your choosing. I recommend you find some place quiet with no distractions.
- You have **three hours to complete the exam.** These must be **consecutive**: once you look at the questions, the clock has started and you must finish within three hours. (You should carry a watch or work in a room with a clock.)

  Exceptions *may* be granted in case of emergency or unforeseen circumstances (e.g., a fire alarm), but it is expected that you turn your cell phone off and let others know that you don't wish to be disturbed.
- You must work **without any assistance**. No books, notes, old homeworks, calculators, Internet, discussing the exam with other people, nothing. If you use a cell phone as a clock, you should put it in airplane mode (i.e. no communications) for the duration of the exam.
- You should bring sufficient blank paper to the exam and write your answers on this.
- Except where noted, you may freely appeal to facts proved in class or in Saracino. You are encouraged to state explicitly what you appeal to – especially if you are not completely sure that you remember correctly. Wrong solutions based on incorrect assumptions will receive more partial credit if you state your assumptions clearly.
- If you find any questions ambiguous, or if you're not sure if your answer is acceptable, explicitly describe your interpretation and/or concerns as part of your solution.
- Please do not discuss the exam with **anyone** until the exams are all turned in.
- **Second Chance:** You may not modify your exam after you are finished. But, if you discover mistakes, you may turn in extra work explaining your mistakes and giving corrected solutions. You can recover up to a quarter of the points that you lost in this way. Be sure to clearly distinguish corrections from your exam.

  For corrections, you may freely refer to your book, your notes, and/or your homeworks, and you may ask me questions (although I don't promise to answer them) – but no talking with anyone else about the exam, and no consulting other sources without permission.
- Please write out the following and sign on the first page of your exam: 'On my honor, I declare that I have followed the rules of this examination'.

- **GOOD LUCK!**

16 points for each question, plus 4 points if you spell your name correctly.

Important: All answers are to be **proved.**

1. Theorem 20.1 of Saracino stated that if F is a field, every ideal of $F[X]$ is principal. Prove this. (You will probably want to use the division algorithm in $F[X]$, which you may take for granted.)

   (Refer to the proof given there for a solution.)

2. Construct a field with 49 elements.

   Let $f(x)$ be any degree 2 polynomial which is irreducible over $\mathbb{Z}_7$. Then, the quotient ring $\mathbb{Z}_7[x]/(f(x))$ is a field extension of $\mathbb{Z}_7$ of degree 2, and hence a field with 49 elements.

   It therefore remains to find such an f. In degree 2, a polynomial is irreducible if and only if it doesn't have any roots, so you need to test this. (You didn't get full credit if you asserted that a polynomial was irreducible without justification.) For example $x^2 + 1$ works – you can plug in each of the 7 values of $\mathbb{Z}_7$ and verfiy that you never get 0.

3. Let $E = \mathbb{Q}(\sqrt[5]{3})$. Find $[E : \mathbb{Q}]$ and find a basis for E over $\mathbb{Q}$.

   Solution. $\sqrt[5]{3}$ is a root of $x^5 - 3$, which is irreducible over $\mathbb{Q}$ by Eisenstein. So $[E : \mathbb{Q}] = 5$, and a basis is given by $\{1, 3^{1/5}, 3^{2/5}, 3^{3/5}, 3^{4/5}\}$.

4. Let $E = \mathbb{Q}(\sqrt[5]{3}, \sqrt{2})$. Find $[E : \mathbb{Q}]$ and find a basis for E over $\mathbb{Q}$.

   Solution. Let $F = \mathbb{Q}(\sqrt[5]{3})$. Then $[F : \mathbb{Q}] = 5$ as seen in the previous problem.

   Now E is generated over F by a root of $x^2 - 2$. So we have $[E : F] = 2$, and therefore $[E : \mathbb{Q}] = [E : F][F : \mathbb{Q}] = 10$, if $x^2 - 2$ is irreducible in E. Otherwise we have $\sqrt{2} \in F$ so that $E = F$ and the answer is the same to that of the previous question.

   We certainly have $\mathbb{Q}(\sqrt{2}) \subseteq E$, and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ (as we know because $x^2 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein). Therefore $[E : \mathbb{Q}]$ is a multiple of 2. So, it must be 10. In particular, $x^2 - 2$ is irreducible over E and an F-basis is given by $\{1, \sqrt{2}\}$.

   A $\mathbb{Q}$-basis for E is given by multiplying the bases for E/F and $F/\mathbb{Q}$:

   $$\{1, 3^{1/5}, 3^{2/5}, 3^{3/5}, 3^{4/5}, \sqrt{2}, \sqrt{2} \cdot 3^{1/5}, \sqrt{2} \cdot 3^{2/5}, \sqrt{2} \cdot 3^{3/5}, \sqrt{2} \cdot 3^{4/5}\}$$

5. Prove that $3^{4/5} + 2 \cdot 3^{1/5} + 4$ is algebraic over $\mathbb{Q}$.

   Note: It is a fact that any finite field extension is algebraic. For other questions you may use this fact freely; for this question, please prove it (or an appropriate special case of it) if you want to apply it.

   Solution. Using the fact to deduce that $\mathbb{Q}(\sqrt[5]{3})$ is algebraic over $\mathbb{Q}$, we just observe that $3^{4/5} + 2 \cdot 3^{1/5} + 4 \in \mathbb{Q}(\sqrt[5]{3})$ – indeed, we have written it explicitly as a linear combination of the basis elements above.

   For a proof of the fact, see Theorem 22.8 of Saracino.

6. Suppose that $F \subseteq E$ and let

$$K = \{a \in E \ : \ a \text{ is algebraic over } F\}.$$

Prove that K is a subfield of E. (K is called the **algebraic closure** of F in E.)

Solution. Given $a, b \in K$, we must prove that $-a$, a^{-1}, $a + b$, and ab are all in K.

These elements are certainly all in $F(a, b)$. Since a is algebraic over F, we have that $F(a)$ is finite over F. Since b is algebraic over F, it is also algebraic over $F(a)$, so that $F(a, b)$ is finite over F. By 'degrees multiply', $F(a, b)$ is finite over F, hence algebraic. In particular all of the above elements are algebraic over F.

7. **Bonus:** Prove that the infinite set $\{2^{1/2}, 2^{1/3}, 2^{1/5}, 2^{1/7}, 2^{1/11}, \cdots\}$, containing $2^{1/p}$ for every prime p, is linearly independent over $\mathbb{Q}$.

(Figure me out at home!!)