Midterm Examination 1 (with partial solutions) - Math 547, Frank Thorne (thorne@math.sc.edu)

Due Monday, March 2 at the beginning of class

Instructions:

- This is a **timed, take-home, closed-book exam.**
- The exam covers Chapters 16 through 18 of Saracino along with the material on the polynomials handout. Understanding the material of Chapter 19 will be helpful but not required.
- You may take the exam at any time and place of your choosing. I recommend you find some place quiet with no distractions.
- You have **three hours to complete the exam.** These must be **consecutive**: once you look at the questions, the clock has started and you must finish within three hours. (You should carry a watch or work in a room with a clock.)

  Exceptions *may* be granted in case of emergency or unforeseen circumstances (e.g., a fire alarm), but it is expected that you turn your cell phone off and let others know that you don't wish to be disturbed.
- You must work **without any assistance**. No books, notes, old homeworks, calculators, Internet, discussing the exam with other people, nothing. If you use a cell phone as a clock, you should put it in airplane mode (i.e. no communications) for the duration of the exam.
- You should bring sufficient blank paper to the exam and write your answers on this.
- Except where noted, you may freely appeal to facts proved in class or in Saracino. You are encouraged to state explicitly what you appeal to – especially if you are not completely sure that you remember correctly. Wrong solutions based on incorrect assumptions will receive more partial credit if you state your assumptions clearly.
- If you find any questions ambiguous, or if you're not sure if your answer is acceptable, explicitly describe your interpretation and/or concerns as part of your solution.
- Please do not discuss the exam with **anyone** until the exams are all turned in.
- **Second Chance:** You may not modify your exam after you are finished. But, if you discover mistakes, you may turn in extra work explaining your mistakes and giving corrected solutions. You can recover up to a quarter of the points that you lost in this way. Be sure to clearly distinguish corrections from your exam.

  For corrections, you may freely refer to your book, your notes, and/or your homeworks, and you may ask me questions (although I don't promise to answer them) – but no talking with anyone else about the exam, and no consulting other sources without permission.
- Please write out the following and sign on the first page of your exam: 'On my honor, I declare that I have followed the rules of this examination'.

- **GOOD LUCK!**

1. (13 points) Consider the set $\mathbb{Q}$, with an addition operation $*$ given by $a * b = a + b - 1$ and a multiplication operation $\square$ given by $a \square b = a + b - ab$. Is $\mathbb{Q}$ with these operations a ring? If so, what is the additive identity? Is it a field? If so, with what multiplicative identity?

   Solution omitted. It's a field, the verification is routine and rather tedious.

2. (13 points) Prove that a field has no nontrivial proper ideals.

   Solution. Suppose that I is a nontrivial proper ideal of a field F. Then I contains a nonzero element x. For any $a \in F$, $\frac{a}{x}$ is also in F and so $x \cdot \frac{a}{x} = a \in I$. Therefore $I = F$, contradicting the assumption that it is proper.

3. (12 points) Let $\phi : R \to S$ be a ring homomorphism. Prove that ϕ is one-to-one if and only if $\mathrm{Ker}(\phi) = \{0_R\}$.

   We must have $0_R \in \mathrm{Ker}(\phi)$, because $\phi(r) = \phi(0_R + r) = \phi(0_R) + \phi(r)$ for any $r \in R$. So $\phi(0_R) = 0_S$, i.e. by definition, $0_R \in \mathrm{Ker}(\phi)$.

   If ϕ is one-to-one then $\mathrm{Ker}(\phi)$ contains only 0_R by definition. Conversely, suppose that $\mathrm{Ker}(\phi) = \{0_R\}$ and that $\phi(r_1) = \phi(r_2)$ for $r_1, r_2 \in R$. Then $0 = \phi(r_1) - \phi(r_2) = \phi(r_1 - r_2)$, so $r_1 - r_2 = 0_R$, i.e. $r_1 = r_2$ so ϕ is one-to-one.

4. (21 points) Let I and J be ideals of a ring R. Prove or disprove (by counterexample) that the following must necessarily be ideals:

   (a) $I \cap J$

   (b) $I \cup J$

   (c) $I + J := \{i + j : i \in I, j \in J\}$

   The first and third are ideals; this is fairly tedious but easy to show and I omit the details. The second is not an ideal; for example let $R = \mathbb{Z}$, $I = (2)$, and $J = (5)$.

5. (21 points) For each condition, give an example of a commutative ring R with unity and an ideal I satisfying the stated condition on I.

   (a) I is maximal,

   (b) I is prime but not maximal,

   (c) I is not prime.

   A maximal ideal is the principal ideal (x) in $\mathbb{R}[x]$. Alternatively, (p) in $\mathbb{Z}$ for any prime p.

   An example of a non-prime ideal is (6) in $\mathbb{Z}$ or (x^2) in $\mathbb{R}[x]$.

   Prime but non-maximal ideals are a little bit harder to find. One example is (x) in $\mathbb{Z}[x]$. Note that this is contained in the larger ideal $(2) + (x)$. An annoying, but correct, example which I didn't think of is (0) in any integral domain which contains a non-trivial ideal. (e.g. $\mathbb{Z}$, $\mathbb{C}[x]$, etc.)

6. (20 points) Consider the map $\phi : \mathbb{R}[x] \to \mathbb{R} \oplus \mathbb{R}$ defined by $\phi(f) = (f(0), f(1))$. Prove that ϕ is a ring homomorphism, compute its kernel, and determine whether or not $\mathbb{R}[x]/\mathrm{Ker}(\phi)$ is an integral domain.

   Brief sketch: Direct computation and application of the definition of a ring homomorphism proves the first part. The kernel is all polynomials which are multiples of both x and $x - 1$ (which is, in fact, the principal ideal $(x^2 - x)$).

   The quotient ring is an integral domain if and only if $\mathrm{Ker}(\phi)$ is a prime ideal. This ideal is not prime, because it contains neither x nor $x - 1$ but it does contain their product.

7. (**Extra Credit.**) (15 points) There are exactly four ring homomorphisms $\mathbb{Q}[x]/(x^3 - 2) \to \mathbb{C}$. Prove this and describe all of them explicitly.