Math 547/702I – Some Homework Solutions

Frank Thorne

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17.6, 7. Find all the maximal ideals in $\mathbb{Z}_n$.

The answer is the set of all ideals $(\overline{p}) = p\mathbb{Z}_n$ where p ranges over prime factors of n. Since $(\overline{p})$ is principal, it is automatically an ideal. To prove maximality, suppose that I is some ideal properly containing $(\overline{p})$. Then I contains an element $\overline{a}$ where a is not a multiple of p. Since we can write $1 = rp + sa$ for some integers r and s, we can also write $\overline{1} = r\overline{p} + s\overline{a}$ in $\mathbb{Z}_n$, so that I is all of $\mathbb{Z}_n$. Therefore $\overline{p}$ is maximal.

Now, how do we know there are no other maximal ideals? Let I be a maximal ideal, and write m for the smallest positive integer such that $\overline{m} \in I$. We have in fact $I = (\overline{m})$. (To see this, suppose that $\overline{r}$ is any element of I not in $\overline{m}$; then we can write $r = km + b$ for some b with $1 \leq b \leq m - 1$ and we see that $\overline{b} \in I$, contradicting minimality.) But then, $I \subseteq (\overline{p})$ for any prime divisor p of m.

Remark. A much nicer proof uses Exercise 18.23 and the remarks after Theorem 18.5, applied to the canonical homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. By this, the maximal ideals in $\mathbb{Z}_n$ are in bijection with the maximal ideals of $\mathbb{Z}$ containing $\mathrm{Ker}(\phi) = n\mathbb{Z}$. The maximal ideals of $\mathbb{Z}$ are all of the form (p) for primes p, and it is easily checked that such an ideal contains (n) if and only if $p \mid n$.

17.16. Show that $M_2(\mathbb{R})$ contains no nontrivial proper ideals.

A complete proof is rather messy, so we just give a sketch. Suppose that I is a nontrivial ideal of $M_2(\mathbb{R})$ and contains some matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where not all of a, b, c, and d are zero.

Suppose that a is not zero. Then, multiply on the right and on the left by matrices which contain a 1 in one entry and zero in the others. By experimentation, you will eventually be able to conclude that $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ are all in I. Now, by multiplying by appropriate diagonal matrices, we can replace a with any number; finally, by adding matrices of these forms we can obtain any matrix in $M_2(\mathbb{R})$.

Now, if b, c, or d is nonzero, a similar proof works – verify! You can also find shortcuts without reinventing the wheel.

17.17. (Sketch.) The other maximal ideal consists of matrices whose left two entries are zero. The proof that this is an ideal, and that it is maximal, is very similar to the proof presented in the book.

Now, suppose you have any proper ideal I of S not contained in one of these two ideals. Then, a and d are both nonzero. Multiplying by $\begin{pmatrix} 1/a & 0 \\ 0 & 1/d \end{pmatrix}$ on the right you obtain that $M_1 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ is in I for some b. If $b = 0$ you're done. (Why?) Otherwise, multiply this matrix by itself to get $M_2 \begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix} \in I$. Now, $M_1 + M_1 - M_2$ is the identity, so now you're done.

(There are other ways to prove this; this is just one.)