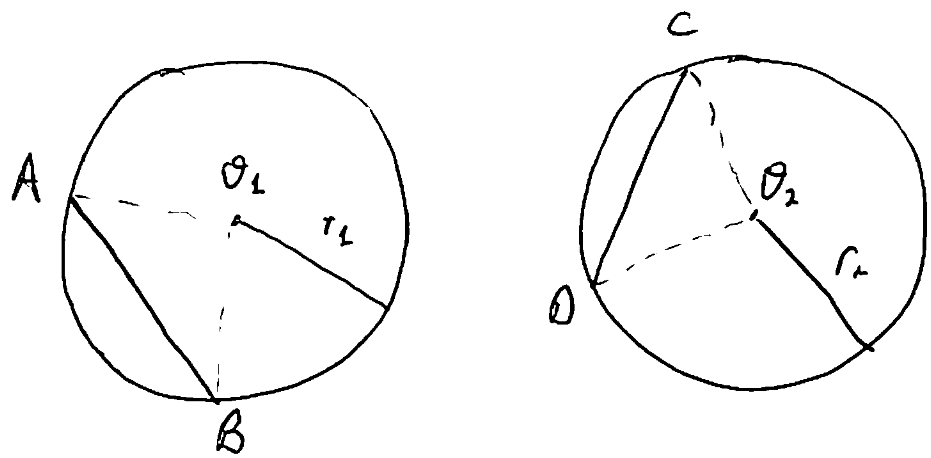
# Solutions 1F.

Problems 1, 2, 6, 7, 10; Challenge: 5, 9, 12, 14.

(1).




Suppose that $r_1 = r_2$. Show: $AB = CD \iff \widehat{AB} \stackrel{\circ}{=} \widehat{CD}$.

**Proof**: ($\Longrightarrow$) Suppose that $AB = CD$. Consider:

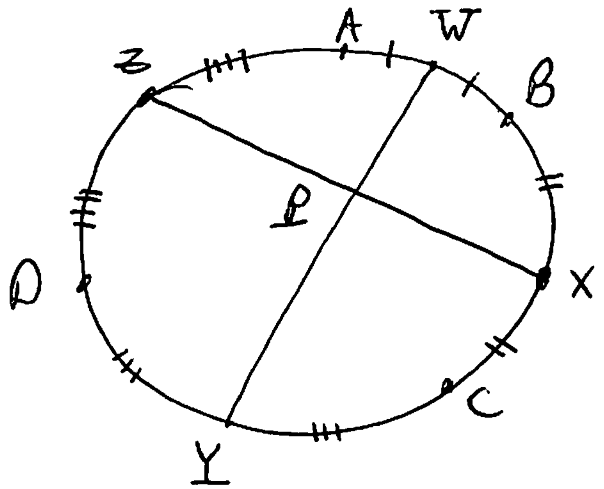
$$\begin{array}{lcl} \underline{\triangle AO_1B} & & \underline{\triangle CO_2D} \\ AO_1 & = r = & CO_2 \\ AB & = & CD \\ O_1B & = & O_2D \end{array} \left.\vphantom{\begin{array}{l} a \\ b \\ c \\ d \end{array}}\right\} \xRightarrow{SSS} \triangle AO_1B \cong \triangle CO_2D$$

$\Longrightarrow \widehat{AB} \stackrel{\circ}{=} \angle AO_1B = \angle CO_2D \stackrel{\circ}{=} \widehat{CD}$ (corr. parts.)

($\Longleftarrow$) Suppose that $\widehat{AB} \stackrel{\circ}{=} \angle AO_1B = \angle CO_2D \stackrel{\circ}{=} \widehat{CD}$.

Consider:
$$\begin{array}{lcl} \underline{\triangle AO_1B} & & \underline{\triangle CO_2D} \\ AO_1 & = r = & CO_2 \\ \angle AO_1B & = & \angle CO_2D \\ O_1B & = r = & O_2D \end{array} \left.\vphantom{\begin{array}{l} a \\ b \\ c \\ d \end{array}}\right\} \begin{array}{l} \xRightarrow{SAS} \triangle AO_1B \cong \triangle CO_2D \\ \Longrightarrow AB = CD \text{ (corr. parts.)} \end{array}$$

(2)




Suppose that: $W, X, Y, Z$ are midpts of $\widehat{AB}, \widehat{BC}, \widehat{CD}, \widehat{DA}$, resp.

Show: $WY \perp XZ$.

Proof: We will show that $\angle WPX = 90°$.

Hypothesis $\Longrightarrow$ $\boxed{\widehat{AW} = \widehat{WB},\ \widehat{BX} = \widehat{XC},\ \widehat{CY} = \widehat{YD},\ \widehat{DZ} = \widehat{ZA}}$ $\ast$

We have: $360° \stackrel{\circ}{=} (\widehat{AW} + \widehat{WB}) + (\widehat{BX} + \widehat{XC}) + (\widehat{CY} + \widehat{YD}) + (\widehat{DZ} + \widehat{ZA})$

$$\text{by } \ast \longrightarrow = 2\widehat{WB} + 2\widehat{BX} + 2\widehat{YD} + 2\widehat{DZ}$$

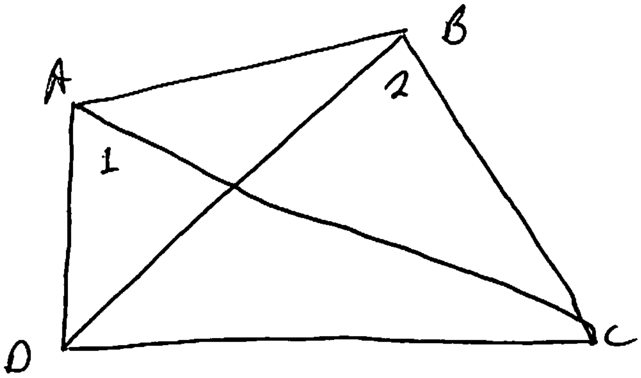
$$= 2(\widehat{WB} + \widehat{BX}) + 2(\widehat{YD} + \widehat{DZ})$$

$$= 2\widehat{WX} + 2\widehat{YZ} = 2(\widehat{WX} + \widehat{YZ})$$

$\stackrel{\div 2}{\Longrightarrow}$ $180° \stackrel{\circ}{=} \widehat{WX} + \widehat{YZ}$. We may now conclude:

$$\angle WPX \underset{\substack{\uparrow \\ \text{Cor. 2.19}}}{\stackrel{\circ}{=}} \frac{1}{2}(\widehat{WX} + \widehat{YZ}) \stackrel{\circ}{=} \frac{1}{2}(180°) = 90°.$$

Hence, we have $WY \perp XZ$.

6. 


Let $ABCD$ be a quadrilateral.

Suppose that $\angle A = 90° = \angle C$.

Show that $\angle DAC = \angle 1 = \angle 2 = \angle DBC$.

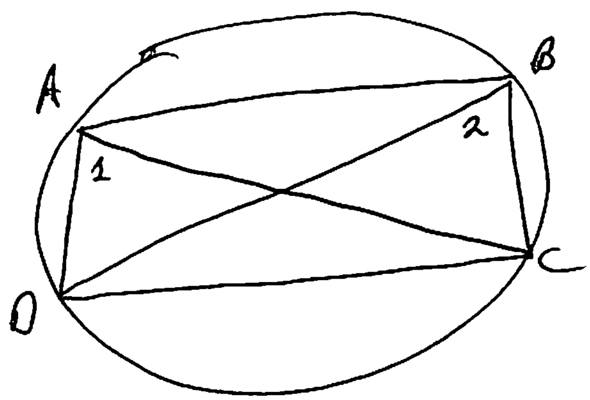
Proof: We first show that $A, B, C, D$ lie on a common circle, circumcircle$(\Delta ABD) = \mathcal{C} =$ circumcircle$(\Delta CBD)$. We have:

$\angle A = 90° \Longrightarrow$ circumcircle$(\Delta ABD)$ has diam. $BD$

$\updownarrow$ 1.22

$\angle C = 90° \Longrightarrow$ circumcircle$(\Delta CBD)$ has diam. $BD$.

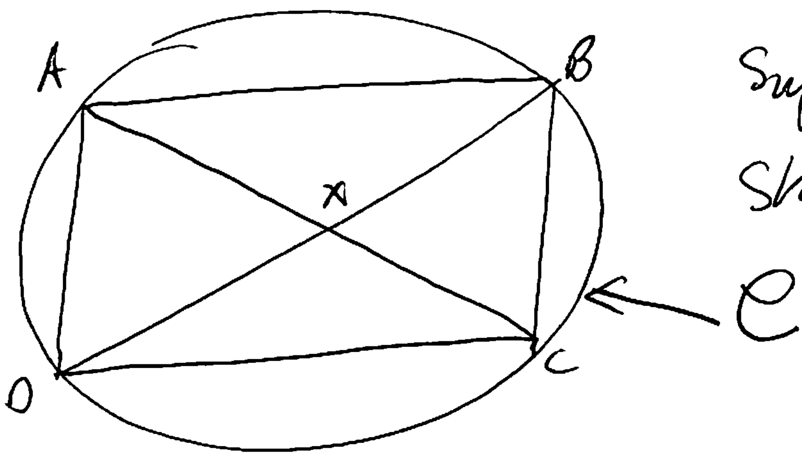
$\Longrightarrow$ the circumcircles are equal (2 circles with the same diam.)

We now have: 


1.16 $\Longrightarrow$

$$\boxed{\angle 1 \overset{\circ}{=} \frac{1}{2}\widehat{DC} \overset{\circ}{=} \angle 2}$$

(7)




Suppose that $AC$ bisects $BD$

Show: $ABCD$ is a rectangle.

__Proof__: Recall that a parallelogram with an interior right angle must be a rectangle. Since $\angle A = 90° = \angle C$, it suffices to show that $ABCD$ is a parallelogram.

For this purpose, we will show that $BD$ bisects $AC$ $(AX = XC)$.

Observe: circle $\mathcal{C}$ has diam $BD$ and $AC$ bisects $BD$ $(BX = XD)$ $\Longrightarrow$ $\mathcal{C}$ has center $X = \text{midpt.}(BD)$.

$$\left.\begin{array}{l}\text{circle } \mathcal{C} \text{ has diam } BD \\ AC \text{ bisects } BD \ (BX = XD)\end{array}\right\} \Longrightarrow \begin{array}{l}\mathcal{C} \text{ has center} \\ X = \text{midpt.}(BD).\end{array}$$

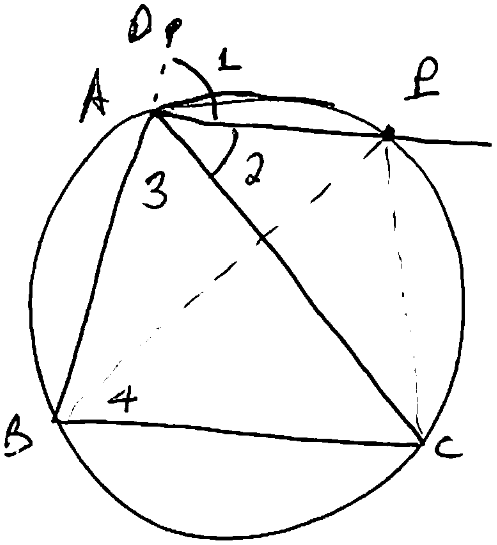
Now, $A$ and $C$ are pts. on $\mathcal{C}$, so $AX$ and $XC$ are radii of $\mathcal{C}$.

Hence, we conclude that $AX = XC$: $BD$ bisects $AC$.

Since $AC$ bisects $BD$ and $BD$ bisects $AC$, we find that $ABCD$ is a parallelogram.

Since it has an interior right angle, it is a rectangle.

(10)




Suppose that $AP = \text{bis}(\angle DAC)$.

Show: $PB = PC$.

<u>Proof</u>: It suffices, by the converse of pons asinorum, to show that $\angle 4 = \angle PCB$. Observe:

$$\left.\begin{array}{l} \angle 2 \overset{\circ}{=} \frac{1}{2}\widehat{PC} \overset{\circ}{=} \angle 4 \\ AP = \text{bis}(\angle DAC) \Rightarrow \angle 1 = \angle 2 \end{array}\right\} \Longrightarrow \angle 1 = \angle 4.$$

Next, note that opp. angles of an inscribed quad. sum to $180^\circ$

$\Longrightarrow \angle 2 + \angle 3 + \angle PCB = 180$. We now have:

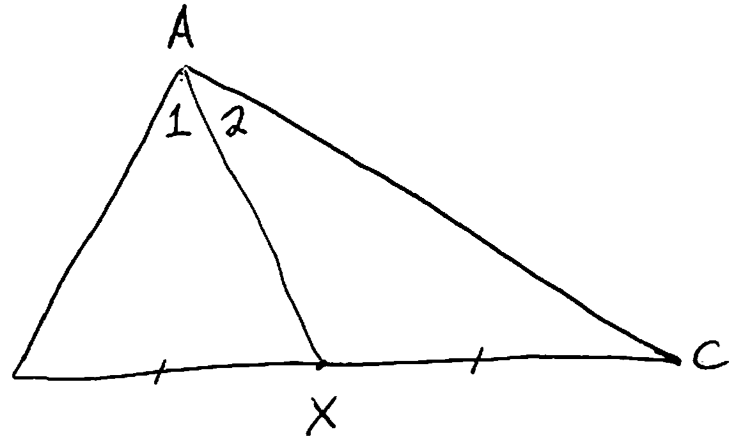
$$\left.\begin{array}{l} \angle 1 + \angle 2 + \angle 3 = 180^\circ \\ \angle PCB + \angle 2 + \angle 3 = 180^\circ \end{array}\right\} \Longrightarrow \angle 1 = \angle PCB.$$

To conclude: $$\left.\begin{array}{l} \angle 1 = \angle 4 \\ \angle 1 = \angle PCB \end{array}\right\} \Longrightarrow \angle 4 = \angle PCB$$

$\Longrightarrow$ $\triangle PBC$ is isosc. (converse of p-a), base $BC$

$\Longrightarrow PB = PC$.

(5)




<u>Claim</u>: Suppose that $AX = \text{med}(A)$. Then we have
$\angle A = 90° \Leftrightarrow AX = \frac{1}{2} BC$.

<u>Proof</u>: First, observe that $AX = \text{med}(A) \Rightarrow BX = XC = \frac{1}{2} BC$

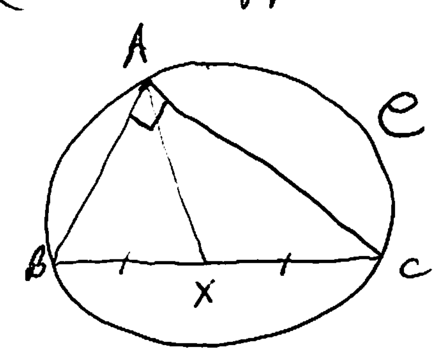
($\Leftarrow$) Suppose that $AX = \frac{1}{2} BC = BX = XC$. It follows that

$$\left.\begin{array}{l} \bullet\ \Delta ABX \text{ is isosceles with base } AB \text{, so } \angle 1 = \angle B \\ \bullet\ \Delta ACX \text{ is isosceles with base } AC \text{, so } \angle 2 = \angle C \end{array}\right\} \text{pons asinorum.}$$

We have: $$\left.\begin{array}{l} \angle A + \angle B + \angle C = 180° \\ \angle A = \angle 1 + \angle 2 \end{array}\right\} \Rightarrow \angle 1 + \angle 2 + \angle B + \angle C = 180°$$

$$\xrightarrow[\angle 2 = \angle C]{\angle 1 = \angle B} 2\angle 1 + 2\angle 2 = 180° \xrightarrow{\div 2} \boxed{\angle A = \angle 1 + \angle 2 = 90°}$$

($\Rightarrow$) Suppose that $\angle A = 90°$. Let $\mathcal{C}$ be the circumcircle of $\Delta ABC$



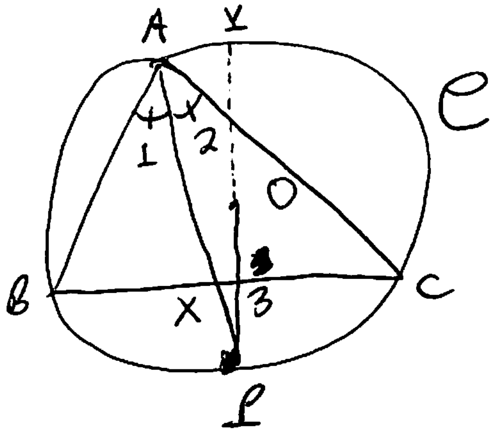

Cor. 1.22: $\angle A = 90° \Rightarrow BC$ is a diameter of $\mathcal{C}$.

$AX = \text{med}(A) \Rightarrow X = \text{midpt}(BC)$
$\Rightarrow X = \text{center of } \mathcal{C}$.

Now, A is on $\mathcal{C} \Rightarrow AX$ is a radius of $\mathcal{C}$. Hence, we have

Radius of $\mathcal{C} = AX = \frac{1}{2}$ diameter of $\mathcal{C} = \frac{1}{2} BC$. I.e, we have $\boxed{AX = \frac{1}{2} BC}$

(9.)




Let $\mathcal{C}$ be the circumcircle of $\Delta ABC$, and let $O$ be the center of $\mathcal{C}$. Suppose that $AX = \text{bis}(\angle A)$, and that $AX$ extends to $P$ on $\mathcal{C}$.

<u>Claim</u>: $OP \perp BC$.

<u>Proof</u>: We have

$$1.16 \Longrightarrow \left\{ \begin{array}{l} \angle 1 \overset{\circ}{=} \frac{1}{2} \widehat{BP} \\ \angle 2 \overset{\circ}{=} \frac{1}{2} \widehat{CP} \end{array} \right.$$

$$AX = \text{bis}(\angle A) \Longrightarrow \angle 1 = \angle 2$$

$$\Longrightarrow \widehat{BP} \overset{\circ}{=} \widehat{CP}.$$

Now, extend $PO$ to $Y$ on $\mathcal{C}$.

$$PY \text{ is a diameter of } \mathcal{C} \Longrightarrow \widehat{BP} + \widehat{BY} \overset{\circ}{=} 180^\circ$$

$$\widehat{BP} \overset{\circ}{=} \widehat{CP}$$

$$\Longrightarrow \widehat{BY} + \widehat{CP} \overset{\circ}{=} 180^\circ$$

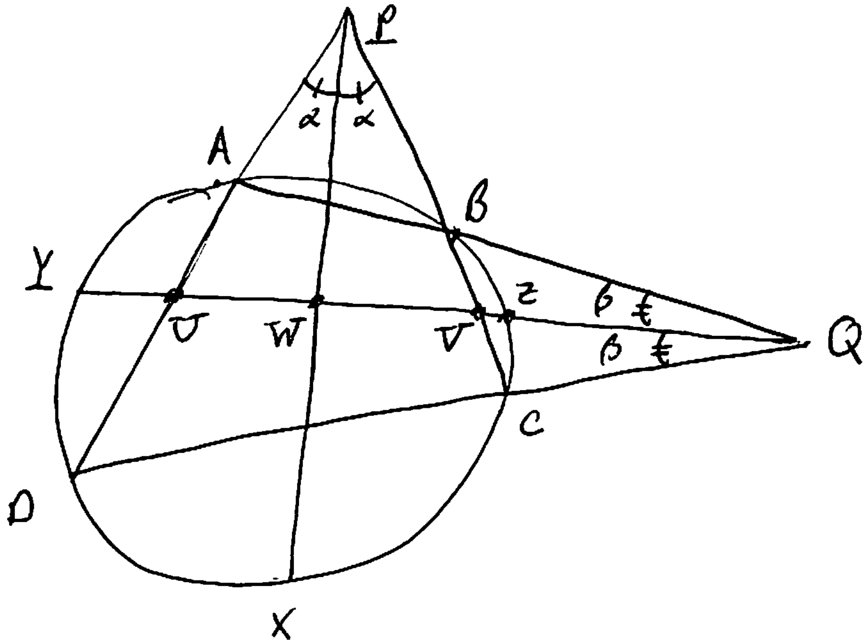
Observe that $1.19 \Longrightarrow \angle 3 \overset{\circ}{=} \frac{1}{2}(\widehat{CP} + \widehat{BY})$

$$180^\circ \overset{\circ}{=} \widehat{CP} + \widehat{BY}$$

$$\Longrightarrow \angle 3 \overset{\circ}{=} \frac{1}{2} \cdot 180^\circ = 90^\circ$$

$$\Longrightarrow \boxed{OP \perp BC}$$

(1 2.)




Suppose that $PX = \text{bis}(\angle P)$ and $QY = \text{bis}(\angle Q)$

<u>Claim</u>: $PX \perp QY$.

<u>Proof</u>: First, note that $PX = \text{bis}(\angle P) \Longrightarrow \angle P = 2\angle\alpha$

$$QY = \text{bis}(\angle Q) \Longrightarrow \angle Q = 2\angle\beta.$$

$$1.18 \Longrightarrow \left\{\begin{array}{l} \angle\beta \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{AY} - \overset{\frown}{BZ}) \\ \angle\beta \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{YD} - \overset{\frown}{CZ}) \end{array}\right\} \Longrightarrow \overset{\frown}{AY} - \overset{\frown}{BZ} \overset{\circ}{=} \overset{\frown}{YD} - \overset{\frown}{CZ}$$

$$\Longrightarrow \underbrace{\overset{\frown}{AY} + \overset{\frown}{CZ} \overset{\circ}{=} \overset{\frown}{YD} + \overset{\frown}{BZ}}$$

But also, $1.19 \Longrightarrow$

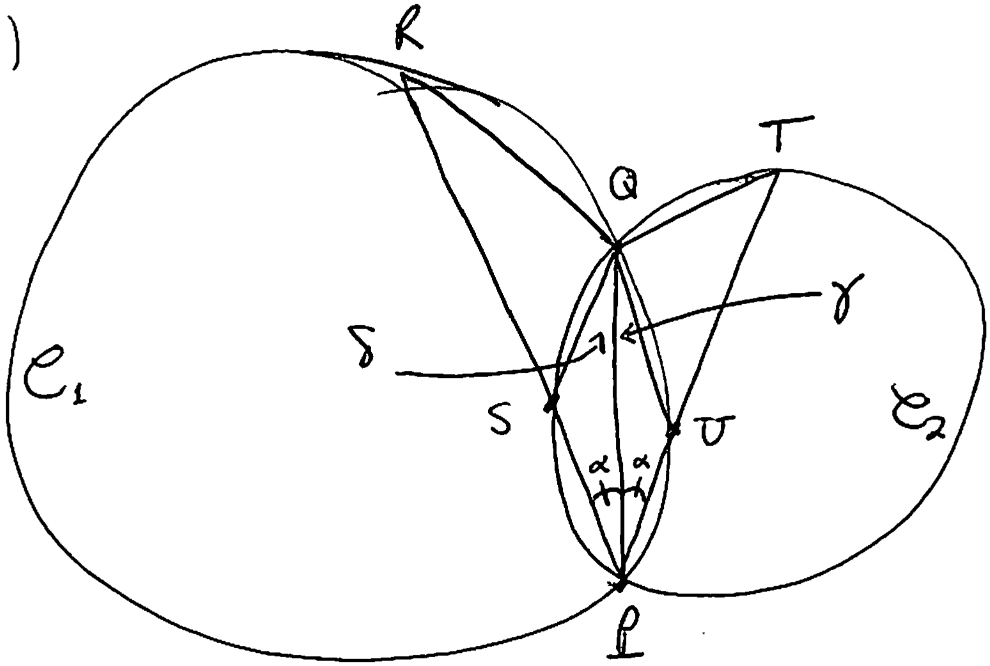
- $\angle U \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{AZ} + \overset{\frown}{YD}) \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{BZ} + \overset{\frown}{YD})$
- $\angle V \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{BY} + \overset{\frown}{CZ}) \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{AY} + \overset{\frown}{CZ}) = \frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{BZ} + \overset{\frown}{YD})$.

Hence, we see that $\angle U = \angle V$. It follows that $\Delta PUV$ is isosceles with base $UV$ and $\text{bis}(\angle P) = PW$

Now, recall 1.2: $\text{bis}(\angle P) = \text{alt}(P)$ in an isosc. triangle with base $UV$.

It follows that $PW \perp UV$ and therefore, that $\boxed{PX \perp QY}$

(14.)




Suppose that $\operatorname{bis}(\angle RPT) = PQ$.

<u>Claim</u>: $RS = TU$.

<u>Proof</u>: $\operatorname{bis}(\angle RPT) = PQ \Longrightarrow \angle RPT = 2\angle\alpha$.

We will show that $\Delta RQS \cong \Delta UQT$. We observe that

$$1.16 \Longrightarrow \left\{\begin{array}{l} \angle\alpha \stackrel{\circ}{=} \frac{1}{2}\widehat{RQ} \\ \angle\alpha \stackrel{\circ}{=} \frac{1}{2}\widehat{QU} \end{array}\right\} \Longrightarrow \widehat{RQ} \doteq \widehat{QU}, \text{ arcs of the same circle, } \mathcal{C}_1$$

$$\xrightarrow{1.FL} \boxed{RQ = QU}$$

Arguing similarly in $\mathcal{C}_2$ using 1.16 and 1F.1, we find that $\boxed{TQ = QS}$

Now, it suffices to show that $\angle TQU = \angle SQR$.

In $\triangle RQS$, we see that

- $\angle S = \angle\alpha + \angle\delta$ (exterior angles)
- $\angle R \overset{\circ}{=} \frac{1}{2}\widehat{QP}$. But also, we have $\left.\begin{array}{l}\angle\gamma \overset{\circ}{=} \frac{1}{2}\widehat{UP} \\ \angle\alpha \overset{\circ}{=} \frac{1}{2}\widehat{QU}\end{array}\right\} \xrightarrow{\text{add}}$

  $\angle\alpha + \angle\gamma \overset{\circ}{=} \frac{1}{2}(\widehat{QU} + \widehat{UP}) \overset{\circ}{=} \frac{1}{2}\widehat{QP}$. It follows that $\angle R = \angle\alpha + \angle\gamma$.
- We conclude that $\angle SQR = \angle Q = 180^\circ - (\angle S + \angle R)$
  $= 180^\circ - (2\angle\alpha + \angle\gamma + \angle\delta)$.

We argue similarly in $\triangle UQT$ to see that

- $\angle U = \angle\alpha + \angle\gamma$ (ext. angles)
- $\angle T = \angle\alpha + \angle\delta$ (similar to computation for $\angle R$ above)
- $\angle TQU = \angle Q = 180^\circ - (2\angle\alpha + \gamma + \angle\delta)$.

Combining facts yields $\boxed{\angle SQR = \angle TQU}$. We now have

$$\begin{array}{ccc} \underline{\triangle RQS} & & \underline{\triangle UQT} \\ RQ & = & QU \\ \angle SQR & = & \angle TQU \\ QS & = & TQ \end{array} \Bigg\} \xrightarrow{SAS} \triangle RQS \cong \triangle UQT$$

$\Longrightarrow \boxed{RS = TU}$ (corresponding parts).