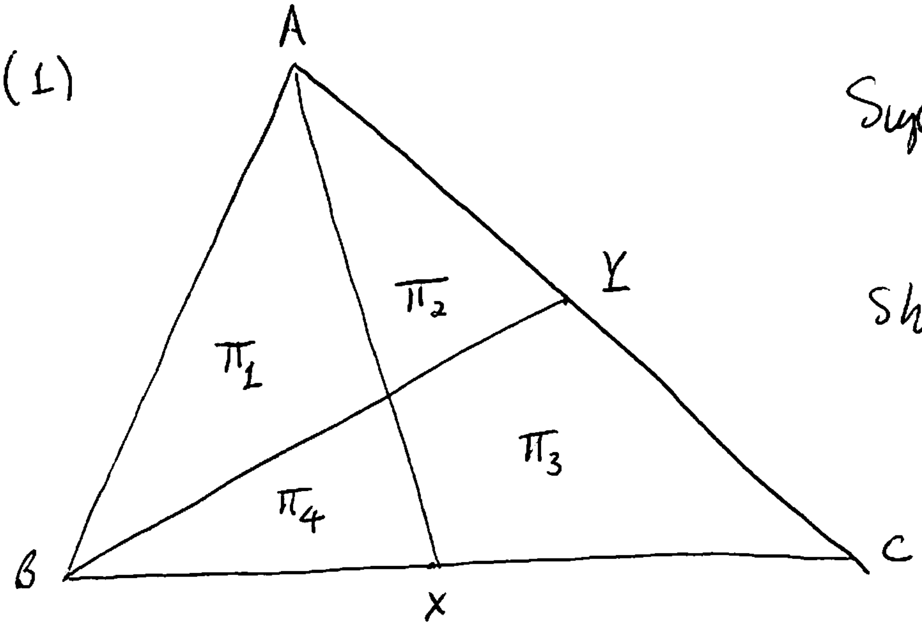
Solutions 1E.

Problems 1, 3.

(1)



Suppose: $AX = \text{med}(A)$

$BY = \text{med}(B)$

Show: $A(\pi_1) = A(\pi_3)$

$A(\pi_2) = A(\pi_4)$

Proof: We first prove that $A(\Delta ABX) = A(\Delta ABY)$.

We have: $AX = \text{med}(A) \Longrightarrow BX = XC \Longrightarrow \underbrace{\tfrac{1}{2}BC = BX}$.

It follows that

$$A(\Delta ABX) = \frac{BX \cdot \text{alt}(A)}{2} = \frac{\tfrac{1}{2}BC \cdot \text{alt}(A)}{2} = \frac{1}{2}\left(\frac{BC \cdot \text{alt}(A)}{2}\right) = \frac{1}{2}A(\Delta ABC)$$

Similarly, $BY = \text{med}(B) \Longrightarrow \underbrace{\tfrac{1}{2}AC = AY}$

$$\Longrightarrow A(\Delta ABY) = \frac{AY \cdot \text{alt}(B)}{2} = \frac{1}{2}\left(\frac{AC \cdot \text{alt}(B)}{2}\right) = \frac{1}{2}A(\Delta ABC)$$

Comparing these yields $A(\Delta ABX) = A(\Delta ABY)$.

Now, observe that $A(\Delta ABY) = A(\pi_1) + A(\pi_2)$

$$A(\Delta ABX) = A(\pi_1) + A(\pi_4).$$

Since $A(\Delta ABY) = A(\Delta ABX)$, we have

$$A(\pi_1) + A(\pi_2) = A(\pi_1) + A(\pi_4) \Longrightarrow \boxed{A(\pi_2) = A(\pi_4)}$$

To conclude, we require $A(\Delta ABX) = A(\Delta ACX)$:

$$A(\Delta ABX) = \frac{1}{2} BX \cdot alt(A) \underset{\substack{\uparrow \\ BX = XC}}{=} \frac{1}{2} XC \cdot alt(A) = A(\Delta ACX)$$

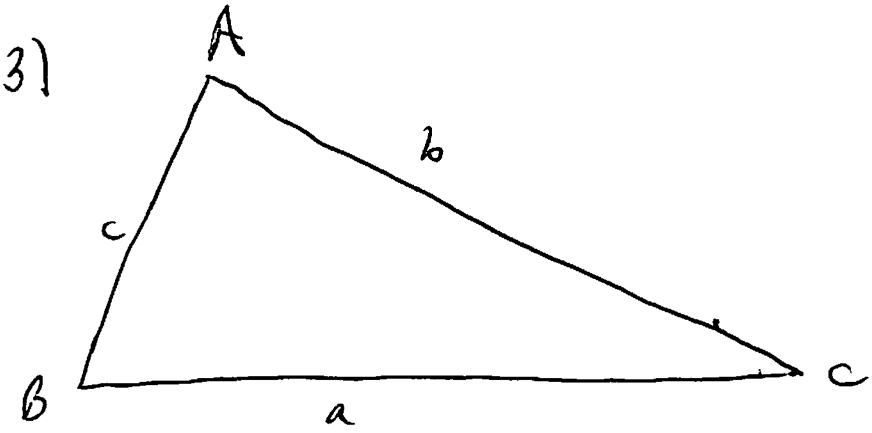
Now, we note that $A(\Delta ABX) = A(\pi_1) + A(\pi_4)$

$$A(\Delta ACX) = A(\pi_2) + A(\pi_3).$$

Since these quantities are equal, and since $A(\pi_2) = A(\pi_4)$,

we have $A(\pi_1) + A(\pi_4) = A(\pi_2) + A(\pi_3) \Longrightarrow \boxed{A(\pi_1) = A(\pi_3)}$

(3)



Give a formula for $A = A(\Delta ABC)$ using only a, B, C.

Solution:

We know that $A = \frac{1}{2} ab \sin C$

Law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B} \Longrightarrow b = \frac{a \sin B}{\sin A}$

$\Longrightarrow A = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin A}$

We also have: $\angle A + \angle B + \angle C = 180^\circ \Longrightarrow \angle A = 180^\circ - (\angle B + \angle C)$

We now conclude: $\boxed{A = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin\left(180^\circ - (B+C)\right)}}$.

(Fact: $\sin(180^\circ - \theta) = \sin\theta$ $\left(= \underbrace{\sin 180 \cos(-\theta)}_{=0} + \underbrace{\cos 180}_{=-1} \underbrace{\sin(-\theta)}_{=-\sin\theta} = \sin\theta\right)$

Hence, we have $\sin(180 - (B+C)) = \sin(B+C)$.

It follows that $\boxed{A = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin(B+C)}}$