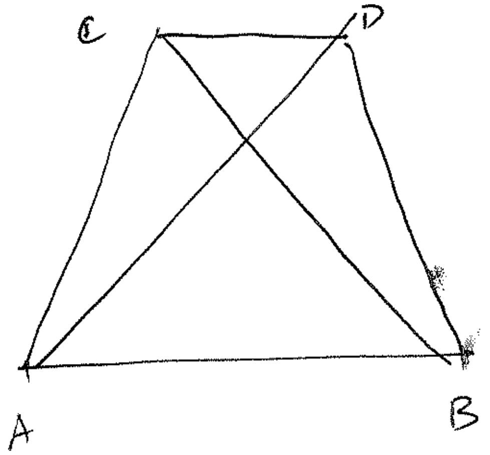
Practice exam solutions.

\#7. Given AB, and straight lines AC and BC, we cannot also have straight lines AD and BD with ~~B~~ C and D on the same side of AB, and AD = AC and BD = BC.



Proof. Suppose we have such straight lines.

Then AC = AD and BC = BD.

Connect CD.

We have

$\angle ACD > \angle BCD$ (whole greater than part)

$= \angle BDC$ (pons asinorum - Prop. 5)

$> ADC$ (whole > part)

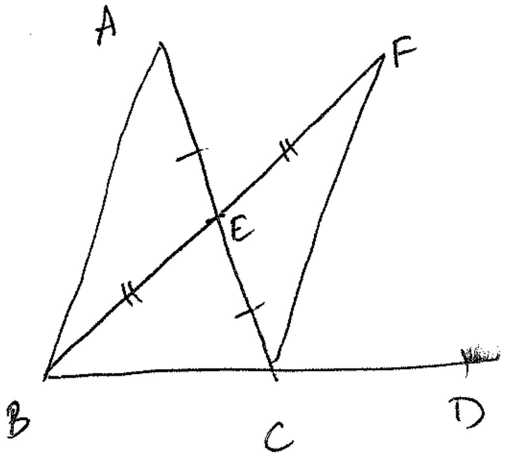
$= \angle ACD$ (pons asinorum)

So $\angle ACD > \angle ACD$ which is absurd.

So such lines cannot exist.

Note. We assumed D was outside $\triangle ABC$ and to the right of C, as Euclid did.

16.



Given a triangle ABC with D as shown.

Bisect AC at E, and extend BE to BF with $BE = EF$. (uses Props 10 and 3)

Then $\angle AEB = \angle CEF$ by vertical angles (Prop. 15)

and $AE = CE$, $BE = EF$,

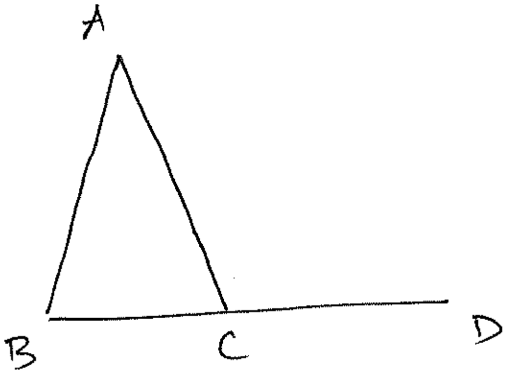
so $\triangle ABE \cong \triangle CFE$ by SAS (Prop 4).

So $\angle BAE = \angle ECF$.

But $\angle ECD > \angle ECF$, so $\angle ECD > \angle BAE$.

Similarly we can show $\angle ECD > \angle ABC$ by bisecting BC instead.

17.



Extend BC to D.

By Prop. 16,

$\angle ACD$ is bigger than $\angle ABC$ or $\angle BAC$.

Adding $\angle BCA$ we see

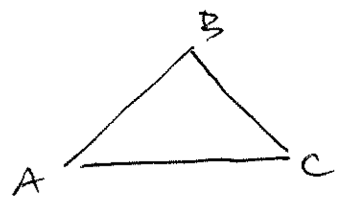
$$\angle ACD + \angle BCA > \angle ABC + \angle BCA$$

$$\angle ACD + \angle BCA > \angle BAC + \angle BCA$$

The left side is $180°$.

So the sum of $\angle BCA$ and either of the other two angles is less than $180°$. We can extend one of the other two sides to prove that the sum of $\angle B$ and $\angle A$ is also $< 180°$.

19. This uses Prop. 18.



Suppose $\angle ABC > \angle BCA$.

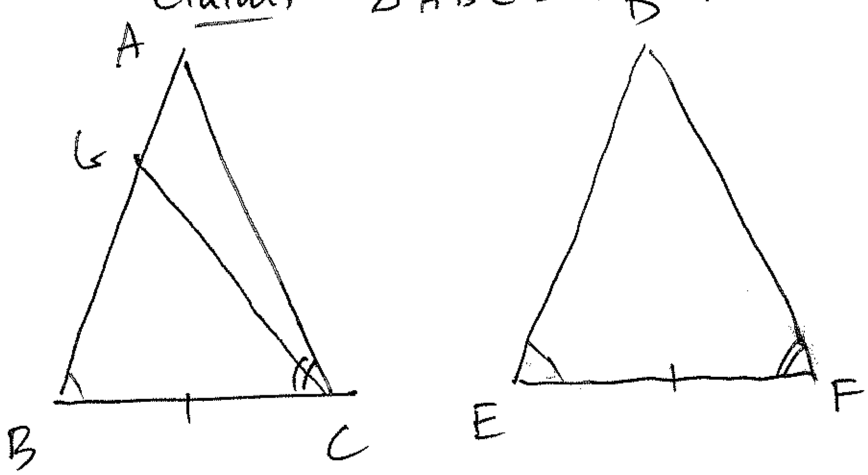
We claim $AC > AB$.

If not, $AC = AB$ or $AC < AB$. The former contradicts the pons asinorum (Prop. 5). If $AC < AB$, then by Prop. 18 $\angle B < \angle C$ which is contrary to what we supposed.

So $AC > AB$.

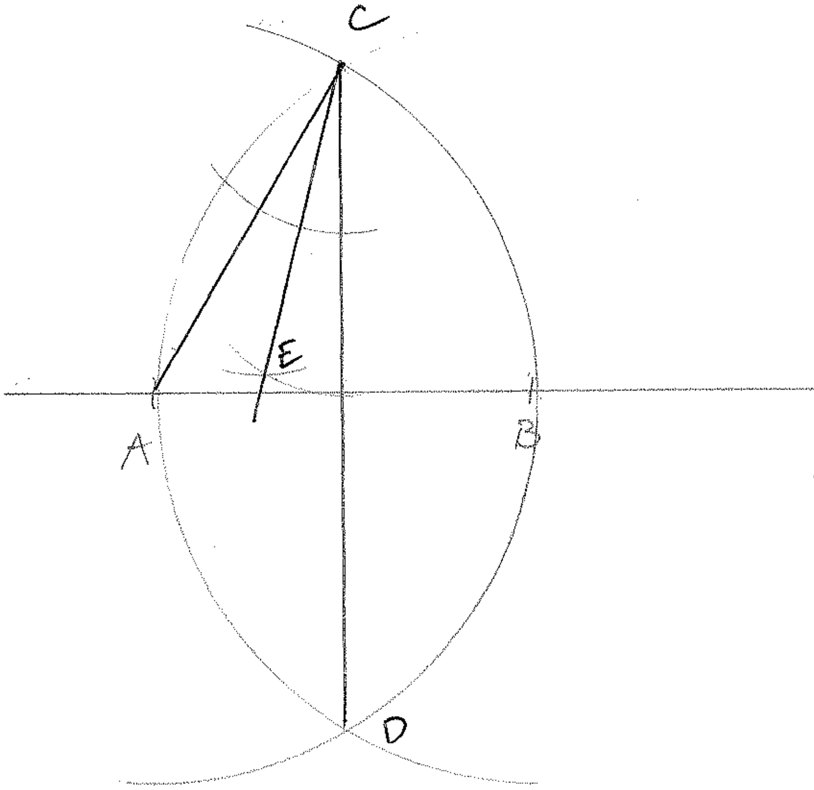
26. Given $\triangle ABC$ and $\triangle DEF$
with $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$,
and $BC = EF$. (we're proving ASA)

Claim. $\triangle ABC \cong \triangle DEF$.



If $AB = DE$ then we're done, by SAS (Prop. 4).
Otherwise, without loss of generality $AB > DE$.
Find G on AB so $BG = DE$.
By SAS (Prop 4) $\triangle GBC \cong \triangle DEF$ so
$\angle GCB = \angle DFE$.
But $\angle GCB < \angle ACB$ (part is less than the whole)
which was assumed equal to $\angle DFE$
which is a contradiction.
So $AB = DE$ and $\triangle ABC \cong \triangle DEF$.

1.



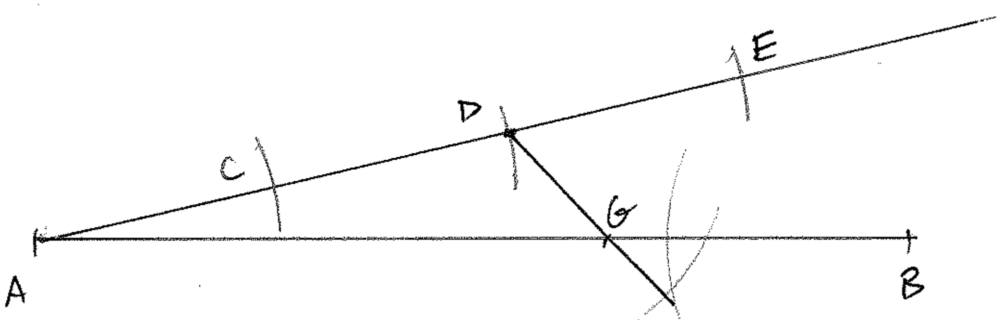
Draw an equilateral triangle ABC on AB as in Euclid's Prop 1.

Let CD be the $\perp$ bisector.

Now let CE be the angle bisector of $\angle ACD$.

$\angle ACE = 15^\circ$.

2.



Given $AB = 1$. Draw another ray from A and mark off three equal segments AC, CD, DE.

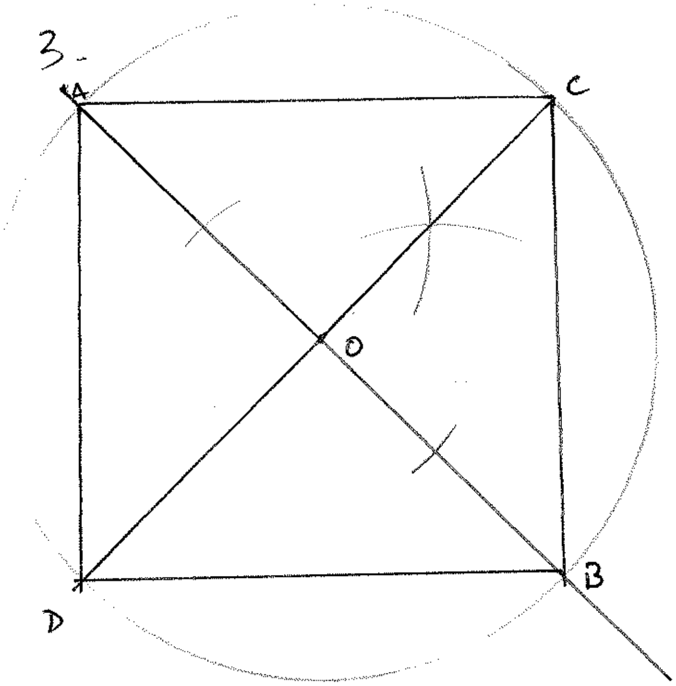
Draw a circle around D of radius EB and around B of radius DE.

These intersect in a point F.

DF intersects AB in a point G.

Then $BG = \frac{1}{3} AB = \frac{1}{3}$.

3.



Let O be the center.
Draw any diameter AB.
Draw the ⊥ to AB at O.
This is a diameter CD.
ACBD is an inscribed square.

4.

Draw any segment AB.
Draw a half circle around A and a full circle around B, both of length AB.
The circles intersect at C and D.
Duplicate the length AC around the circle. Connect the dots to get a regular hexagon.