Quiz 8 - Math 374, Frank Thorne (thorne@math.sc.edu)

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A sequence is defined recursively by $s_0 = 3$, and $s_k = s_{k-1} + 2k$ for all integers $k \geq 1$. Guess, and then prove (by mathematical induction) an explicit formula for the sequence s_n.

Hint: Look for a formula of the form $s_n = n^2 + *$, *where* $*$ *is something simple.*

Solution.

Work out some small values of s_n: $s_0 = 3$, $s_1 = 5$, $s_2 = 9$, $s_3 = 15$, $s_4 = 23$, $s_5 = 33$, and so on. Following the hint, and comparing these to n^2, we see that each s_n is $n+3$ larger than n^2. In other words, $s_n = n^2 + n + 3$.

We now prove this for all $n \geq 0$. The base case says that $s_0 = 0^2 + 0 + 3 = 3$, which is true. So, assume that $s_k = k^2 + k + 3$ for an arbitrary integer $k \geq 0$. We must prove that $s_{k+1} = (k+1)^2 + (k+1) + 3$.

We have that

$$s_{k+1} = s_k + 2(k+1) = k^2 + k + 3 + 2(k+1) = k^2 + 3k + 5,$$

by (in order): the recursive definition of s_k, the inductive hypothesis, and algebra.

But we have also that

$$(k+1)^2 + (k+1) + 3 = (k^2 + 2k + 1) + (k+1) + 3 = k^2 + 3k + 5.$$

Therefore, $s_{k+1} = (k+1)^2 + (k+1) + 3$ as desired. Hence tne result follows for all integers $n \geq 0$ by induction.