Quiz 7 - Math 374, Frank Thorne (thorne@math.sc.edu)

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Prove, using mathematical induction, that

$$5^n + 9 < 6^n,$$

for all integers $n \geq 2$.

Solution. First of all, observe that $5^2 + 9 < 6^2$, because this just says that $34 < 36$.

Now, suppose that $5^k + 9 < 6^k$ for an arbitrary integer $k \geq 2$. We want to prove that $5^{k+1} + 9 < 6^{k+1}$, and then this establishes the proof of the statement for all $n \geq 2$ by induction.

The idea is that the function on the right is growing faster than that on the left.

Proof 1. We have

$$5^{k+1} + 9 = 5 \cdot 5^k + 9 < 5 \cdot (6^k - 9) + 9,$$

by using the inductive hypothesis in the alternative form $5^k < 6^k - 9$. Now, we have

$$5 \cdot (6^k - 9) + 9 = 5 \cdot 6^k - 45 + 9 = 5 \cdot 6^k - 36.$$

But, $5 \cdot 6^k < 6 \cdot 6^k = 6^{k+1}$ and $-36 < 0$, so that this quantity is less than 6^{k+1}. Therefore, $5^{k+1} + 9 < 6^{k+1}$, as desired.

Proof 2. Given that $5^k + 9 < 6^k$, add $4 \cdot 5^k$ to both sides, so that we get $5 \cdot 5^k + 9 < 6^k + 4 \cdot 5^k$, i.e. $5^{k+1} + 9 < 6^k + 4 \cdot 5^k$. But,

$$6^k + 4 \cdot 5^k < 6^k + 5 \cdot 6^k = 6 \cdot 6^k = 6^{k+1},$$

so that $5^{k+1} + 9 < 6^k$, as desired.

Other variations are possible as well. There is no one right way to prove this fact, and a little bit of creativity is required. One advantage you have is that the inequality to be proved is true by a large margin — so that you have quite a bit of 'wiggle room', so to speak.