Quiz 6 - Math 374, Frank Thorne (thorne@math.sc.edu)

Monday, March 2, 2015

Prove, using mathematical induction, that

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2,$$

for all integers $n \geq 0$.

Solution. For $n = 0$, the statement says that

$$\sum_{i=1}^{1} i \cdot 2^i = 0 \cdot 2^{0+2} + 2,$$

i.e., that $2 = 2$, which is true.

Now, let $k \geq 0$ be an arbitrary positive integer and assume that the statement is true for $n = k$. Then, by rewriting the sum we have

$$\sum_{i=1}^{(k+1)+1} i \cdot 2^i = \left(\sum_{i=1}^{k+1} i \cdot 2^i \right) + (k+2) \cdot 2^{k+2}.$$

(The part outside the big parentheses is the $i = (k+1) + 1 = k + 2$ term.)

We use the inductive hypothesis to rewrite the first sum on the right, and obtain

$$\sum_{i=1}^{(k+1)+1} i \cdot 2^i = k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2}.$$

Upon using algebra to rearrange the right hand side, we see that we have

$$\begin{aligned} \sum_{i=1}^{(k+1)+1} i \cdot 2^i &= (k + k + 2) \cdot 2^{k+2} + 2 \\ &= (2k+2) \cdot 2^{k+2} + 2 \\ &= 2(k+1) \cdot 2^{k+2} + 2 \\ &= (k+1) \cdot 2^{k+3} + 2 \\ &= (k+1) \cdot 2^{(k+1)+2} + 2. \end{aligned}$$

Therefore, the statement is true for $n = k + 1$, so that the result follows by induction.

(Makeup problem on next page)

Makeup problem (March 4, 2015)

Prove by induction that $7^n - 2^n$ is divisible by 5 for each integer $n \geq 0$.

Solution. For $n = 0$, $7^n - 2^n = 1 - 1 = 0$ which is divisible by 5.
So assume, for an arbitrary integer $k \geq 0$, that $7^k - 2^k$ is divisible by 5. Then,

$$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k = (5 + 2) \cdot 7^k - 2 \cdot 2^k.$$

Using algebra to rearrange, this equals

$$5 \cdot 7^k + 2 \cdot (7^k - 2^k).$$

By assumption, $7^k - 2^k$ is divisible by 5. Therefore $2 \cdot (7^k - 2^k)$ is also divisible by 5, and since $5 \cdot 7^k$ is a multiple of 5 the entire expression is divisible by 5. Therefore the result follows for $k + 1$, and hence for all $n \geq 0$ by induction.