Final Examination - Math 374, Frank Thorne (thorne@math.sc.edu)

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Please work without books, notes, calculators, or any assistance from others. Unless otherwise stated, justification is required for full credit.

(1) (6 points, justification not required) "If unobtanium is boiling, then its temperature must be at least 5000 degrees." Given that this is true, which of the following also must be true?

(a) Unobtanium will boil only if its temperature is at least 5000 degrees.

(b) If unobtanium is not boiling, then its temperature is less than 5000 degrees.

(c) A sufficient condition for unobtanium to boil is that its temperature be at least 5000 degrees.

(d) If the temperature of unobtanium is less than 5000 degrees, then it is not boiling.

(2) (9 points, justification not required) For each of the following statements:

- Formally write it using appropriate truth variables, predicates, domains, and quantifiers. (e.g. your answer might look like $p \to \sim q$ or $\forall x \in D, P(x) \to Q(x)$ where you say what p, q, D, P, Q are.)
- Using either English or logical notation (your choice), write a negation of the statement. *For quantified statements, your negation should be such that the innermost statement is negated. (For example, do not write 'There is no ...' in response to 'There is an...'.)*

(a) All dogs are loyal.

(b) If the Gamecocks won their football game, then the offense stopped the blitz and the defense stopped the running game.

(c) There is a student who has done every homework exercise.

(3) (6 points) Use truth tables to determine whether or not the argument form is valid. Explain your answer.

$$r \to \sim q$$
$$r \vee q$$
$$r \to p$$
$$\therefore p$$

(4) (6 points) A set of premises and a conclusion are given. Use the valid argument forms given in the attached table to deduce the conclusion from the premises. (Cite each argument form by name when you use it.)

(a) $\sim q \to u \wedge s$

(b) $q \to r$

(c) $w \wedge s \to p$

(d) $w \vee q$

(e) $\sim r$

(f) $\therefore\ p$

(5) (8 points) Prove, using mathematical induction, that for each integer $n \geq 1$, that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(6) (8 points) Prove, using mathematical induction, that for each integer $n \geq 0$, that $5^n - 1$ is divisible by 4.

(7) (9 points) Suppose that h_k is a sequence defined by $h_0 = 1$, and

$$h_k = 2^k - h_{k-1} \text{ for all integers } k \geq 1.$$

Determine an explicit formula for h_k and prove it using mathematical induction.

(8) (5 points) You throw two ordinary dice. The sample space can be written

$$S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, \cdots, 61, 62, 63, 64, 65, 66\},$$

where the first and second digits indicate the number on the first and second die respectively. (The $\cdots$ stands for the 18 elements where the first die is 3, 4, or 5.)

Let E be the event that you throw a total of at least 9. Write E as a subset of S (justification not required), and compute its probability.

(9) (8 points) In a certain state, all automobile license plates have four letters followed by three digits.

(a) How many different license plates are possible?

(b) How many license plates could begin with an A and end with a 0?

(c) How many license plates could begin with AB and have all letters and numbers distinct?

(d) Suppose that 20 of the four-letter combinations are considered obscene words. How many license plates are possible which avoid these? *(The restrictions of the previous parts no longer hold.)*

(10) (5 points) 50 students are asked about which beverages they enjoy:

- 21 of them like coffee.
- 21 of them like tea.
- 31 of them like cola.
- 9 like both coffee and tea.

- 14 like both coffee and cola.
- 15 like both tea and cola.
- 41 like at least one of coffee, tea, and cola.

How many of them like all three?

(11) (12 points)

(a) How many distinguishable ways can the letters of the word HULLABALOO be arranged in order?

(b) How many distinguishable orderings of the letters of HULLABALOO begin with U and end with L?

(c) How many distinguishable orderings of the letters of HULLABALOO contain the two letters HU next to each other in order?

(12) (12 points) A student council consists of 10 students, with five men and five women: Ann, Bob, Cal, Dee, Eve, Flo, Gus, Hal, Ina, and Jed.

For this question you must simplify your answer completely; i.e., do not give answers such as $9 \cdot 8 \cdot 7$ *or* $\binom{8}{5}$.

(a) In how many ways can a committee of four be selected from the membership of the council?

(b) In how many ways can a committee of four be selected if Flo refuses to serve?

(c) In how many ways can a committee of four be selected if Flo and Bob insist on both serving, if either is to serve?

(d) In how many ways can a committee of four be selected if it must have at least two women?

(13) (6 points) Determine whether or not the graph below has an Euler circuit. If it does not, explain why not. If it does, describe one.

(14) (Extra Credit. 10 points.) Recall that the Ackermann function is defined, for all nonnegative integers m and n, by

$$A(m,n) = \begin{cases} n+1 \text{ if } m = 0, \\ A(m-1,1) \text{ if } m > 0 \text{ and } n = 0, \\ A(m-1, A(m,n-1)) \text{ if } m > 0 \text{ and } n > 0. \end{cases}$$

Prove that $A(4,4) \geq 10000000000000000000000000000000$.

1. (6) (a) Yes (b) No (c) No (d) Yes

2. (9) (a) $\forall x \in D \quad L(x)$ where D = dogs

$L(x)$ = "L is loyal"

Negation: $\exists x \in D \quad \sim L(x)$

There is a dog which is not loyal.

(b) $W \longrightarrow B \wedge R$,

where W = "The Gamecocks won their football game"

B = "The offence stopped the blitz"

R = "The defense stopped the running game"

Negation: $\cancel{(B \wedge R) \wedge \sim W}$ wrong!!!

~~The offence stopped the blitz, and the defense sto~~

$W \wedge (\sim(B \wedge R))$

$W \wedge (\sim B \vee \sim R)$

The Gamecocks won, but either the offence didn't stop the blitz, or the defence did not stop the running game.

(c) $\exists x \in D \quad \forall y \in E \quad P(x,y)$

D = students

E = exercises

$P(x,y)$ = "x has done y"

Negation: $\forall x \in D \quad \exists y \in E \quad \sim P(x,y)$

For every student there is an exercise they haven't done.

16)

p	q	r	$\sim q$	$r \to \sim q$	$r \vee q$	$r \to p$	p
T	T	T	F	F	T	T	T
T	T	F	F	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	F	T	T
F	T	T	F	F	T	F	F
F	T	F	F	T	T	T	F
F	F	T	T	T	T	F	F
F	F	F	T	T	F	T	F

In the circled row the premises are all true and the conclusion is not.
So the argument is not valid.

4.16)

$q \to r$
$\sim r$
$\therefore \sim q$ (modus tollens)

$\sim q \to u \wedge s$
$\sim q$
$\therefore u \wedge s$ (modus ponens)

$w \vee q$
$\sim q$
$\therefore w$ (elimination)

$u \wedge s$
$\therefore s$ (specialization)

w
s
$\therefore w \wedge s$ (conjunction)

$w \wedge s \to p$
$w \wedge s$
$\therefore p$ (modus ponens)

5. (8) The base case says $1^2 = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$

which is true.

Assume $1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$

for arbitrary $k \geq 1$.

Then $1^2 + \cdots + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$$= \frac{(k^2+k)(2k+1)}{6} + \frac{6k^2+12k+6}{6}$$

$$= \frac{2k^3+3k^2+k+6k^2+12k+6}{6}$$

$$= \frac{2k^3+9k^2+13k+6}{6}.$$

Now $\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k^2+3k+2)(2k+3)}{6}$$

$$= \frac{2k^2+9k^2+13k+6}{6}$$

These are equal so the result follows by induction.

6. (8) For $n=0$ this says $5^0 - 1 = 1 - 1 = 0$ is divisible by 4.

Assume for some $k \geq 0$ that $5^k - 1$ is divisible by 4.

Then
$$5^{k+1} - 1 = 5 \cdot 5^k - 1$$
$$= (4+1)5^k - 1$$
$$= 4 \cdot 5^k + 5^k - 1$$

Now $4 \cdot 5^k$ and $5^k - 1$ are both divisible by 4, so $5^{k+1} - 1$ is too.

The result follows by induction.

7. (9) $h_0 = 1$, $h_1 = 2 - 1 = 1$, $h_2 = 4 - 1 = 3$,
$$h_3 = 8 - 3 = 5,$$
$$h_4 = 16 - 5 = 11$$
$$h_5 = 32 - 11 = 21$$
$$h_6 = 64 - 21 = 43$$
$$h_7 = 128 - 41 = 87$$

Revised problem, with $h_k = 2^k + h_{k-1}$:

$h_0 = 1$, $h_1 = 3$, $h_2 = 7$, $h_3 = 15$ etc.

Claim. $h_n = 2^{n+1} - 1$ for all $n \geq 0$.

Proof. For $n = 0$, true because $1 = 2^1 - 1$.

If $h_k = 2^{k+1} - 1$, then

$$h_{k+1} = 2^{k+1} + h_k \quad \text{(by def.)}$$
$$= 2^{k+1} + 2^{k+1} - 1 \quad \text{(by induction)}$$
$$= 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$$

So the result follows by induction.

8. (5) $E = \{36, 45, \cancel{46} 54, 63, 46, 55, 64, 56, 65, 66\}$

$$\text{Prob.} = \frac{|E|}{|S|} = \frac{10}{36} = \frac{5}{18}.$$

$\{36, 45, 54, 63\}: 2,$

9. (8)

(a) All choices are independent: $26^4 \cdot 10^3$.

(b) Can only choose middle five: $26^3 \cdot 10^2$.

(c) 24 choices for ~~first~~ third letter
23 choices for fourth
10, 9, 8 for digits respectively $\quad 24 \cdot 23 \cdot 10 \cdot 9 \cdot 8$

(d) Number with obscene words: $20 \cdot 10^3$
(any combo of digits allowed)

So number of allowed plates is
$$26^4 \cdot 10^3 - 20 \cdot 10^3.$$
$[-20 : -1]$

10. (5) Let K = students who like coffee

T = " " tea

C = " " cola.

By Inclusion - Exclusion

$$|K \cup T \cup C| = |K| + |T| + |C| - |K \cap T| - |K \cap C| - |T \cap C| + |K \cap T \cap C|$$

$$41 = 21 + 21 + 31 - 9 - 14 - 15 + |K \cap T \cap C|$$

$$= 73 - 38 + |K \cap T \cap C|$$

$$= 35 + |K \cap T \cap C|$$

so $|K \cap T \cap C| = 6$.

11. (a) There are 2 A's and 3 L's, 2 O's, no other repeats.

(12)

$$\frac{10!}{2!\,3!\,2!}$$

(b) Equivalent to arranging HLABALOO.

Similarly to above, $\frac{8!}{2!\,2!\,2!}$

(c) Count "HU" as one letter, so $\frac{9!}{2!\,3!\,2!}$

12.

12 (a) $\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 3 \cdot 7 = 210.$

(b) 9 left to choose from.

$\binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 3 \cdot 7 \cdot 6 = 126.$

(c) Number with both: Choose 2 out of the 8 remaining.

$\binom{8}{2} = \frac{8 \cdot 7}{2 \cdot 1} = 28$

Number with neither:

$\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 2 \cdot 5 = 70.$

So $28 + 70 = 98.$

(d) Number with one woman:

$\binom{5}{1}\binom{5}{3} = 5 \cdot 10 = 50$

((Choose woman, then the men)

With none:

$\binom{5}{0}\binom{5}{4} = 1 \cdot 5 = 5.$

So $210 - (50 + 5) = 155.$

13. It does:

16 $r \to z \to y \to x \to w \to y \to u \to z \to s \to u \to w \to v \to u \to t \to s \to r.$