Midterm Examination 2 - Math 374, Frank Thorne (thorne@math.sc.edu)

Wednesday, March 27, 2015

Please work without books, notes, calculators, or any assistance from others.

There are five questions. In each case give a proof by mathematical induction – regular induction, strong induction, or structural induction as appropriate. Your *best four solutions* will be worth 25 points each and your worst will be thrown out.

(1) Prove, for all integers $n \geq 1$, that

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Solution. Let $P(n)$ be the statement $\sum_{i=1}^{n}(2n-1) = n^2$. We prove $P(n)$ for all $n \geq 1$ by induction.

The base case $P(1)$ is true because

$$\sum_{i=1}^{1}(2i-1) = 2 \cdot 1 - 1 = 1 = 1^2.$$

Now, suppose that $P(k)$ is true for an arbitrary $k \geq 1$. We have

$$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + (2(k+1)-1).$$

Using the inductive hypothesis, this equals

$$k^2 + (2(k+1)-1) = k^2 + (2k+2-1) = k^2 + 2k + 1.$$

Since this equals $(k+1)^2$, $P(k+1)$ is true, so that the result follows by induction.

(2) Prove, for all integers $n \geq 2$, that

$$3 + 5n \leq 4^n.$$

Solution. Let $P(n)$ be the statement $3 + 5n \leq 4^n$. The base case $n = 2$ is true because it says that $13 \leq 16$.

Now, suppose that $P(k)$ is true for an arbitrary $k \geq 2$. We must prove $P(k+1)$. In other words, given that $3 + 5k \leq 4^n$, we must prove that $3 + 5(k+1) = 8 + 5k \leq 4^{k+1}$.

We are adding 5 to the left side and $3 \cdot 4^k$ to the right side. Since $k \geq 2$, we have $3 \cdot 4^k \geq 3 \cdot 16 = 48$, so we have $5 \leq 3 \cdot 4^k$. Therefore,

$$8 + 5k = 5 + (3 + 5k) \leq 3 \cdot 4^k + 4^k,$$

where we used the above algebra and the inductive hypothesis to give upper bounds for 5 and $3+5k$ respectively. Since the right side of this expression equals 4^{k+1}, the result follows for all $k \geq 2$ by induction.

Alternate solution. We skip to the proof of $P(k+1)$ (the rest of the solution is the same as the first solution). Multiplying the inductive hypothesis $3+5k \leq 4^k$ by 4 we see that $12+20k \leq 4^{k+1}$. We know that

$$3+5(k+1) = 8+5k \leq 12+20k \leq 4^{k+1},$$

so that $P(k+1)$ is true.

(3) Suppose that $a_0, a_1, a_2, \cdots$ is a sequence defined by

$$a_0 = 5, a_1 = 16, \text{ and } a_k = 7a_{k-1} - 10a_{k-2} \text{ for all integers } k \geq 2.$$

Prove, for all integers $n \geq 0$, that

$$a_n = 3 \cdot 2^n + 2 \cdot 5^n.$$

Solution. We have two base cases to check. We have that

$$3 \cdot 2^0 + 2 \cdot 5^0 = 3 + 2 = 5 = a_0,$$

$$3 \cdot 2^1 + 2 \cdot 5^1 = 6 + 10 = 16 = a_1.$$

Now, assume that the equality holds for all i with $0 \leq i \leq k$; by strong induction the equality holds for all $n \geq 0$ if we can prove it for $k+1$.

We have

$$3 \cdot 2^{k+1} + 2 \cdot 5^{k+1} = 3 \cdot 2 \cdot 2^k + 2 \cdot 5 \cdot 5^k = 6 \cdot 2^k + 10 \cdot 5^k,$$

and we want to prove that a_{k+1} equals this. We have, by definition,

$$a_{k+1} = 7a_k - 10a_{k-1}.$$

Applying the inductive hypothesis for both a_k and a_{k-1} (this is the strong induction step) we have

$$a_{k+1} = 7(3 \cdot 2^k + 2 \cdot 5^k) - 10(3 \cdot 2^{k-1} + 2 \cdot 5^{k-1}) = 21 \cdot 2^k + 14 \cdot 5^k - 30 \cdot 2^{k-1} - 20 \cdot 5^{k-1}.$$

Since $2^{k-1} = \frac{1}{2}2^k$ and $5^{k-1} = \frac{1}{5}5^k$ we have

$$a_{k+1} = 21 \cdot 2^k + 14 \cdot 5^k - 15 \cdot 2^k - 4 \cdot 5^k = 6 \cdot 2^k - 10 \cdot 5^k,$$

the same expression as above. So the result follows by induction.

(4) Define a sequence by $a_1 = 2$ and $a_{k+1} = 2a_k - 1$ for all $k \geq 1$. Guess, and then prove, an explicit formula for a_n which is valid for all $n \geq 0$.

Hint: The sequence is approximately doubling at each step. This information might inform a preliminary guess, which you can then adjust.

Solution. We write out the first few terms. $a_1 = 2$, $a_2 = 3$, $a_3 = 5$, $a_4 = 9$, $a_5 = 17$. We see that these are one higher than powers of two, which is consistent with the hint. (The sequence would be powers of two if it were *exactly* doubling at each step.) The powers of two are offset by 1 and by a little bit of trial and error we write down $a_n = 1 + 2^{n-1}$.

To prove this by induction, note first that $a_1 = 2 = 1 + 2^{1-1}$. Now, assume the formula holds for $n = k$ for some $k \geq 1$. Then,

$$a_{k+1} = 2a_k - 1 = 2(1 + 2^{k-1}) - 1 = 2 + 2^k - 1 = 2^k + 1,$$

so that the result holds by induction.

(5) A set S of arithmetic expressions is given by the following recursive definition.

- (Base) Any number is in S. (By a number I mean something like 73 or π, but an expression like $5 * 8$ which has to be computed first.)
- (Recursion) If A and B are in S, then $(A + B)$ is in S.
- (Recursion) If A and B are in S, then $(A * B)$ is in S.
- (Restriction) Nothing else is in S.

So, for example, a typical element of S looks something like

$$(((7 * 9) * (3 + 2)) + 5).$$

Prove that any arithmetic expression in S contains more numbers than plus signs.

Solution. We do this by structural induction. In the base case, any number contains one number and no plus signs. So certainly the statement to be proved holds.

Now, assume that A and B are in S and have more numbers than plus signs. We must prove that $(A + B)$ and $(A * B)$ both have more numbers than plus signs.

Suppose that A has m numbers and B has n numbers. Then, by the inductive hypothesis, A has at most $m - 1$ plus signs and B has at most $n - 1$ plus signs. $(A + B)$ has $m + n$ numbers (the total number in A and B), and it has at most $(m - 1) + (n - 1) + 1 = m + n - 1$ plus signs (the total number in A and B, plus the one connecting them). So, $(A + B)$ has more numbers than plus signs, as desired.

Similarly, $(A * B)$ has $m + n$ numbers and at most $m + n - 2$ plus signs (because this time they are not connected with a plus sign.) So, $(A * B)$ has more numbers than plus signs, as desired.

Since the base case is proved, and the inductive step is proved for each of the two pieces of the recursive definition, the result follows by structural induction.