Midterm Examination 1 - Math 374, Frank Thorne (thorne@math.sc.edu)

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Please work without books, notes, calculators, or any assistance from others. You don't need to justify your answers except where noted; however, wrong answers without any justification will receive little or no credit.

(1) (12 points)

(a) Is $4 = \{4\}$?

No. 4 is a number. $\{4\}$ is a set containing 4, which is something different.

(b) How many elements are in the set $\{3, 4, 3, 5\}$?

3. This is the same as $\{3, 4, 5\}$ – duplicates don't count twice.

(c) How many elements are in the set $\{1, \{1\}, \{1, \{1\}\}\}$?

3. The three elements are 1, $\{1\}$, and $\{1, \{1\}\}$.

(2) (12 points) For each sentence below, say whether it is logically equivalent to the sentence

If it's snowing, the temperature must be at or below freezing.

(a) If the temperature is above freezing, then it is not snowing.

Logically equivalent. This is the contrapositive of the original statement.

(b) It only snows if the temperature is at or below freezing.

Logically equivalent. This is the same as 'It's snowing only if the temperature is at or below freezing', and 'X only if Y' means the same thing as 'If X, then Y'. (See p. 45.)

(c) A necessary condition for it to snow is that the temperature be at or below freezing.

Logically equivalent. 'A necessary condition for X is Y' means the same as 'If X, then Y'.

(3) (16 points) Formally rewrite each of the following statements using appropriate truth variables, predicates, and quantifiers. (e.g. your answer might look like $p \to \sim q$ or $\forall x \in D, P(x) \to Q(x)$ where you say what p, q, D, P, Q are.)

(a) Math is hard but it's fun.

Let p be 'math is hard' and q be 'math is fun'. Then: $p \wedge q$

(b) If it walks like a duck and it quacks like a duck, then it is a duck. (Treat this as an *implicitly quantified* statement; i.e., your answer should have a quantifier.)

Let $P(x)$ be 'x walks like a duck', $Q(x)$ be 'x quacks like a duck', and $R(x)$ be 'x is a duck.' Let the domain D be the set of animals (other possibilities for D also exist). Then the best answer is

$$\forall x \in D,\ (P(x) \wedge Q(x)) \rightarrow R(x).$$

Note that it is not completely correct to write that 'Q is 'it quacks like a duck''. What is 'it'? It is important to use variables precisely in predicates, for the same reason that it is in computer programs.

Also, letting D be the set of ducks is not a suitable choice of domain. We think of this statement as applying to some larger category of animals – either all animals (or 'everything') or at least animals which are seen walking or swimming around ponds.

(c) Any integer with an even square is even.

Let $P(x)$ be 'x has an even square' and $Q(x)$ be 'x is even'. Then,

$$\forall x \in \mathbb{Z},\ P(x) \rightarrow Q(x).$$

Some people wrote

$$\forall x \in \mathbb{Z},\ Q(x^2) \rightarrow Q(x).$$

Very strictly speaking, this is not correct, because the language of predicate logic does not allow for arithmetic inside these statements, but I also accepted this as a correct answer.

(d) Some real numbers are rational.

Let $P(x)$ be 'x is rational'; then

$$\exists x \in \mathbb{R},\ P(x).$$

Other solutions are also possible. For example, let D be the domain of 'all numbers' (such a concept is not well defined, but never mind) and $Q(x)$ be 'x is real'; then one solution is

$$\exists x \in D,\ P(x) \wedge Q(x).$$

(4) (12 points) Determine whether the statement forms

$$(p \wedge q) \vee r \quad \text{and} \quad p \wedge (q \vee r)$$

are logically equivalent. Construct truth tables for these statement forms (which you may combine into one truth table if you wish) and justify your conclusion.

This is p. 38, Problem 23; please see the solution in the back of the book. (Almost everyone got this correct.)

(5) (15 points) Write negations of the following statements. For quantified statements, your negation should be such that the innermost statement is negated. (For example, do not write 'There is no ...' in response to 'There is an...'.)

(a) If P is a square, then P is a rectangle.

Negation: P is a square and P is not a rectangle.

(b) There is a computer which gives the correct answer to every question which is posed to it.

Negation: Every computer gives an incorrect answer to some question which is posed to it.

(c) For all positive real numbers x, there is a positive real number y such that $xy = 1$.

Negation: There is a real number x such that for all real numbers y we have $xy \neq 1$.

These all follow the recipe! Please reread the appropriate section of the book if you had difficulties.

(6) (10 points) Use truth tables to determine whether or not the argument form is valid. Explain your answer.

$$\begin{array}{l} p \to q \\ q \to p \\ \therefore p \vee q \end{array}$$

This is Chapter 2.3, number 6. (Indeed it is on the same page as the list of argument forms!) Please see the back of your book for a solution. Note that it doesn't suffice to write out the truth table and then say that the argument form is invalid. The question asked you to explain why.

(7) (13 points) A set of premises and a conclusion are given. Use the valid argument forms given in the attached table to deduce the conclusion from the premises. (Cite each argument form by name when you use it.)

(a) $\sim p \vee q \to r$

(b) $s \vee \sim q$

(c) $\sim t$

(d) $p \to t$

(e) $\sim p \wedge r \to \sim s$

(f) $\therefore \sim q$

This is Chapter 2.3, number 41. Please see the back of your book for a solution.

The most common way that people missed points is by omitting steps. For example, if you know that $\sim p$ and $\sim p \vee q \to r$ and you want to conclude r, this takes two steps: first conclude $\sim p \vee q$ by generalization.

This is not a severe mistake and it was not severely penalized, but since the list of argument forms was provided to you it was expected that you would adhere to them strictly.

(8) (10 points) Determine whether the arguments are below are valid or invalid. Justify your answers – using English explanations, a diagram, and/or by citing the rule of inference used or type of error.

(a) Nothing intelligible ever puzzles me. Logic puzzles me. Therefore, logic is unintelligible.

This inference is valid. Some people successfully explained this using English or using a diagram. You can also regard it as an instance of universal modus tollens by writing $P(x)$ for 'x is intelligible', $Q(x)$ for 'x puzzles me', and expressing this argument as

$$\begin{array}{r} \forall x, P(x) \rightarrow \sim Q(x) \\ Q(L) \\ \therefore\ P(L) \end{array}$$

(b) All freshmen must take writing. Caroline is a freshman. Therefore, Caroline must take writing.

Valid my universal modus ponens. Pretty straightforward.