Structural induction.

Defs. A base and a recursion (with a restriction)

Ex.
Def. A parenthesis structure is:

(base: 1) (), or

(rec: 2a) (E), where E is a P.S., or

(rec: 2b) EF, where E, F are P.S.'s.

Proofs by induction:
Prove for all cases of the definition.

Ex. ~~Prove that any point in a parenthesis structure,~~
~~between two symbols,~~

Let $f(E, n)$ = # open parentheses in first n symbols of E
— # closed parens in first n symbols of E.

e.g. $E = (()())$ what is $f(E, n)$?

Claim. For all P and n, $f(E, n) \geq 0$.

Proof. Base. If $P = ()$, $f(E, 0) = 0$
$f(E, 1) = 1$
$f(E, 2) = 0$.

Recursion. If $P = (E)$, then (let n = length of P)

$$f(P, 0) = 0 \geq 0$$
$$f(P, n) = 0 \geq 0$$

For $1 \leq k \leq n-1$,
$$f(P, k) = 1 + f(E, k-1)$$
$$\geq 1. \text{ (by induction.}$$

If $P = EF$, let r = length of E, s = length of F.

$f(P, k) = f(E, k)$ when $0 \leq k \leq r$

$f(P, k) = f(F, k-r)$ when $r \leq k \leq r+s$.

Ill - defined recursive functions.

$$G(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 + G\left(\frac{n}{2}\right) & \text{if } n \text{ even} \\ G(3n-1) & \text{if } n \text{ odd},\ n > 1. \end{cases}$$

Compute $G(1)$ through $G(5)$.

Def. of $\mathbb{Z}^+$:

A <u>positive</u> <u>integer</u> is:

(base) 1, <u>or</u>

(recursion) $n+1$, where n is a positive integer.

THIS IS WHY INDUCTION WORKS.

<u>Review.</u>

5.1. sequences and Σ-notation.

5.2-3. Intro to induction.

Examples. Any amount of money $\geq 8¢$ from 3, 5¢ coins.

Identities. $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{n}\right) = \frac{1}{n}.$$

Divisibility. For all $n \geq 0$, $2^{2n} - 1$ div by 3

$n^3 - 7n + 3$ " "

$7^n - 2^n$ div by 5.

Inequalities. $1 + 3n \leq 4^n$ for all $n \geq 0$.

$2^n < (n+1)!$ for $n \geq 2$.

Strong induction. Good for more complicated

e.g. $e_0 = 12$, $e_1 = 29$

$e_k = 5e_{k-1} - 6e_{k-2}$ for all integers $k \geq 2$

Prove $e_n = 5 \cdot 3^n + 7 \cdot 2^n$ for $n \geq 0$.

5.5 (skip)

5.6 Recursively defined sequences.

F_k : $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

$C_1 = 1$ and $C_n = \frac{4n-2}{n+1} C_{n-1}$.

Then $C_n = \frac{(2n)!}{n!(n+1)!}$

and $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$.

Tower of Hanoi : write the recursion yourself!

5.7. Guessing recursions

Sometimes easy. In any case, look for patterns.

(5.8 - More sophisticated guessing)

5.9. Structural.