Quiz 5 - Math 374, Frank Thorne (thorne@math.sc.edu)

Friday, September 29, 2017

(1) Prove that $2^n > n^2$ for all $n \geq 5$.

Proof. We prove this by induction. The base case is the statement that $2^5 > 5^2$, which is true because $32 > 25$.

To prove the base case, we assume that $2^k > k^2$ for some $k \geq 5$, and prove that $2^{k+1} > (k+1)^2$. There are several ways to do this.

Proof #1. We have $2^{k+1} = 2^k + 2^k > 2k^2$, by the inductive hypothesis. So we need to prove that $2k^2 > (k+1)^2$, or $2k^2 > k^2 + 2k + 1$.

Since $k \geq 5$, we have $2k + 1 < 3k < k^2$, and wo $k^2 + 2k + 1 < k^2 + k^2 = 2k^2$, as desired.

Proof #2. We have $2^{k+1} = 2^k + 2^k > 2k^2$, by the inductive hypothesis. So we need to prove that $2k^2 > (k+1)^2$, which we rewrite as $\frac{(k+1)^2}{k^2} < 2$.

We have

$$\frac{(k+1)^2}{k^2} = \frac{k^2 + 2k + 1}{k^2} = 1 + \frac{2}{k} + \frac{1}{k^2}.$$

Since $k \geq 5$, we have $\frac{2}{k} < \frac{1}{2}$ and $\frac{1}{k^2} < \frac{1}{2}$, so this whole quantity is less than 2, as desired.

There are other possibilities too!