Structural induction.

Defs. A base and a recursion (with a restriction)

Ex. Def. A parenthesis structure is:
(base: 1) $()$, or
(rec: 2a) $(E)$, where $E$ is a P.S., or
(rec: 2b) $EF$, where $E, F$ are P.S.'s.

Proofs by induction:
Prove for all cases of the definition.

Ex. Prove that any point in a parenthesis structure
between two symbols
Let $f(E, n) = \# \text{ open parentheses in first } n \text{ symbols of } E$
$\# \text{ closed parentheses in first } n \text{ symbols of } E$.

E.g. $E = (())(1)$ what is $f(E, 2)$?

Claim. For all $E$ and $n$, $f(E, n) \geq 0$.

Proof. Base. If $E = (1)$, $f(E, 0) = 0$
$f(E, 1) = 1$
$f(E, 2) = 0$.

Recursion. If $P = (E)$, then (let $n =$ length of $P$)
$f(P, 0) = 0 \geq 0$
$f(P, n) = 0 \geq 0$
For $1 \leq k \leq n - 1$,
$f(P, k) = 1 + f(E, k - 1)$
$\geq 1$. (by induction.)

If $P = EF$, let $r =$ length of $E$, $S =$ length of $F$.
$f(P, k) = f(E, k)$ when $0 \leq k \leq r$
$f(P, k) = f(F, \lceil k - r \rceil)$ when $r < k \leq n$. 

Ill-defined recursive functions.

\[
G(n) = \begin{cases} 
1 & \text{if } n = 1 \\
1 + G\left(\frac{n}{2}\right) & \text{if } n \text{ even} \\
G(3n - 1) & \text{if } n \text{ odd}, \ n > 1.
\end{cases}
\]

Compute \( G(1) \) through \( G(5) \).

Definition of \( \mathbb{Z}^+ \):

A positive integer is:

(base) \( 1 \), or

(reCURSion) \( n+1 \) where \( n \) is a positive integer.

This is why induction works.

Review:

5.1. Sequences and \( \Sigma \)-notation.

5.2-3. Intro to induction.

Examples. Any amount of money \( \geq 8 \) from 2, 5, \( \text{dimes} \) coins.

Identities. 
\[
1 + 2 + \cdots + n = \frac{n(n+1)}{2}
\]

\[
1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.
\]

\[
\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{n}\right) = \frac{1}{n}.
\]

Divisibility. For all \( n \geq 0 \), \( 2^n - 1 \) div. by 3

\[
u^3 - 7n + 3
\]

\( 7^n - 2^n \) div. by 5.

Inequalities. 
\( 1 + 3n \leq 4^n \) for all \( n \geq 0 \).

\( 2^n \leq (n+1)! \) for \( n \geq 2 \).
Strong induction. Good for more complicated

e.g. \( e_0 = 12, \ e_1 = 29 \)
\( e_k = 5e_{k-1} - 6e_{k-2} \) for all integers \( k \geq 2 \)
Prove \( e_n = 5 \cdot 3^n + 7 \cdot 2^n \) for \( n \geq 0 \).

5.5 (skip)

5.6 Recursively defined sequences.
\( F_k : \ F_0 = F_1 = 1, \ F_n = F_{n-1} + F_{n-2} \) for all \( n \geq 2 \).
\( C_1 = 1 \) and \( C_n = \frac{4n-2}{2n+1} \) \( C_{n-1} \).

Then \( C_n = \frac{(2n)!}{n!n+1} \)!

and \( F_n = \sqrt{\phi} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right] \).

Tower of Hanoi: write the recursion yourself!

5.7 Guessing recursions

Sometimes easy. In any case, look for patterns.
(5.8 - More sophisticated guessing)
5.9 Structural