Midterm Examination 1 - Math 374, Frank Thorne (thorne@math.sc.edu)

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Please work without books, notes, calculators, or any assistance from others. **Exception:** You may rely on the list of rules of inference provided to you.

(1) (16 points) Construct a truth table for the wff $A \wedge B \to A'$. If it is a tautology or contradiction, say so.

Solution. (1.1, 17a)

A	B	$A \wedge B$	A'	$A \wedge B \to A'$
T	T	T	F	F
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

The statement is neither a tautology nor a contradiction, since it is true for some truth values of A and B, and false for others.

(2) (20 points) Introducing statement variables, formulate the argument in symbolic form, and use propositional logic to prove that the argument is valid.

If the ad is successful, then the sales volume will go up. Either the ad is successful or the store will close. The sales volume will not go up. Therefore the store will close.

Solution. (1.2, 44) Write S for 'the ad is successful'; V for 'the sales volume will go up'; C for 'the store will close'. The argument can be formulated as

$$(S \to V) \wedge (S \vee C) \wedge V' \to C.$$

Multiple proofs can be given. Perhaps the most straightforward is:

1. $S \to V$, (hypothesis)
2. $S \vee C$, (hypothesis)
3. V' (hypothesis)
4. S' (1, 3, modus tollens)
5. C (2, 4, elimination)

(3) (12 points each) Using predicate symbols and appropriate quantifiers, write each English language statement as a predicate wff. In addition, write the negation of each statement (in English).

You may use the following quantifiers and predicates: $L(x)$ is 'x is a lion'; $P(x)$ is 'x is a predator'; $Z(x)$ is 'x is a zebra'; $E(x, y)$ is 'x eats y'. The domain consists of all animals.

(a) All lions are predators.

Solution. (1.3, 12a)

$$\forall x L(x) \to P(x).$$

A negation is 'Some lions are not predators'.

(b) Some lions eat all zebras.

Solution. (1.3, 12d) The simplest solution is

$$\exists x L(x) \wedge (\forall y Z(y) \to E(x,y).$$

'There is at least one lion so that, for every other animal, if it is a zebra then this lion eats it.'

A negation is 'All lions don't eat some zebras'. Or, equivalently, 'For every lion, there are some zebras it doesn't eat.'

(4) (20 points) Prove, for every positive integer n, that

$$1 + 5 + 5^2 + \cdots + 5^{n-1} = \frac{5^n - 1}{5 - 1}.$$

Solution. (2.2, 12) We prove this by induction. The base case asserts that

$$1 = \frac{5^1 - 1}{5 - 1}.$$

This is true since $5^1 = 5$.

Now, assume for some $k \geq 1$ that

$$1 + 5 + 5^2 + \cdots + 5^{k-1} = \frac{5^k - 1}{5 - 1}.$$

We will be done if we can prove that

$$1 + 5 + 5^2 + \cdots + 5^k = \frac{5^{k+1} - 1}{5 - 1}.$$

To prove this, note that we have

$$\begin{aligned}
1 + 5 + 5^2 + \cdots + 5^k &= \frac{5^k - 1}{5 - 1} + 5^k \\
&= \frac{5^k - 1}{5 - 1} + \frac{(5-1)5^k}{(5-1)} \\
&= \frac{5^k - 1 + (5-1)5^k}{5 - 1} \\
&= \frac{(1 + 5 - 1)5^k - 1}{5 - 1} \\
&= \frac{5 \cdot 5^k - 1}{5 - 1} \\
&= \frac{5^{k+1} - 1}{5 - 1},
\end{aligned}$$

as desired. (The first step used the inductive hypothesis, and the rest is algebra.)

(5) (20 points) Recall that the Fibonacci numbers $F(n) = 1$ are defined by $F(1) = 1$, $F(2) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$.

Prove, for each $n \geq 3$, that $F(n+1) + F(n-2) = 2F(n)$.

It is possible to give a proof by induction, or to give a proof directly from the definition. Either is fine.

Solution #1. (2.4, 11) We prove this by induction. We prove two base cases: $n = 3$ and $n = 4$. We have $F(1) = 1, F(2) = 1, F(3) = 2, F(4) = 3, F(5) = 5$.

The case $n = 3$ asserts that $F(4) + F(1) = 2F(3)$, which says that $3 + 1 = 2 \cdot 2$, which is true.

The case $n = 4$ asserts that $F(5) + F(2) = 2F(4)$, which says that $5 + 1 = 2 \cdot 3$, which is true.

Now, assume, for some fixed $k \geq 5$, that $F(n+1) + F(n-2) = 2F(n)$ for all $n < k$. We will prove that $F(k+1) + F(k-2) = 2F(k)$.

By definition, we have

$$F(k+1)+F(k-2) = (F(k)+F(k-1))+(F(k-3)+F(k-4)) = (F(k)+F(k-3))+(F(k-1)+F(k-4)).$$

By the inductive hypothesis applied to $n = k-1$, we have $F(k)+F(k-3) = 2F(k-1)$. By the inductive hypothesis applied to $n = k-2$, we have $F(k-1)+F(k-4) = 2F(k-2)$. Therefore, we have

$$F(k+1) + F(k-2) = 2F(k-1) + 2F(k-2) = 2F(k),$$

using the definition of the Fibonacci numbers in the last step, and we're done.

Solution #2. Here is a (much easier!) proof directly by the definition. We have, because $n \geq 3$,

$$F(n+1) + F(n-2) = (F(n) + F(n-1)) + F(n-2) = F(n) + (F(n-1) + F(n-2)) = 2F(n).$$