3. The series \( 3 + 2 + \frac{4}{3} + \frac{6}{9} + \ldots \) is a geometric series with the first term \( a = 3 \) and common ratio \( r = \frac{2}{3} \). Because \(|r| < 1\), it converges and the value is

\[
\frac{a}{1 - r} = \frac{3}{1 - \frac{2}{3}} = \frac{3}{\frac{1}{3}} = 9.
\]

4. The series \( \sum_{n=1}^{\infty} \frac{1}{n^2 + q} \) converges if and only if \( \int_1^{\infty} \frac{1}{x^2 + q} \, dx \) does, by the integral test.

\[
\cot(\theta) = \frac{x}{3}, \quad \sin(\theta) = \frac{3}{\sqrt{x^2 + q}}.
\]

So: \( x = 3 \cot(\theta) \),

\[
dx = 3 \cot(\theta) \, d(\theta) = -3 \csc^2(\theta) \, d(\theta).
\]

This is why it is good to use the quotient rule to check!

\[
\sin^2(\theta) = \frac{q}{x^2 + q}, \quad \sin^2(\theta) = \frac{1}{x^2 + q}.
\]

and \( \int_1^{\infty} \frac{1}{x^2 + q} \, dx = \int_{\theta = 1}^{\theta = \infty} \frac{\sin^2(\theta)}{q} \cdot (-3 \csc^2(\theta) \cdot \frac{1}{\sin^2(\theta)} \, d(\theta)) \)

\[
= \frac{1}{3} \left[ -\cos \theta \right]_{\theta = 1}^{\theta = \infty} = \left[ -\frac{1}{3} \theta \right]_{x = 1}^{x = \infty} \]

\[
= \frac{1}{3} \cot^{-1} \left( \frac{x}{3} \right) \bigg|_{x = 1}^{x = \infty}.
\]
\[ \lim_{y \to \infty} \frac{-1}{3} \cot^{-1} \left( \frac{y}{3} \right) + \frac{1}{3} \cot^{-1} \left( \frac{1}{3} \right). \]

What is \( \cot^{-1}(810) \)?

\[ \cot \theta = \frac{adj}{opp} \]

Here \( \cot \theta \) is very big.

Drawing a graph or triangle lets us recall that

\[ \lim_{x \to \infty} -\frac{1}{3} \cot^{-1} \left( \frac{x}{3} \right) = 0. \]

In particular this integral converges, and so does our sum to \( \frac{1}{3} \cot^{-1} \left( \frac{1}{3} \right) \).

Alternate solution:

If we drew the triangle differently, would get

\[ \frac{1}{3} \tan^{-1} \left( \frac{x}{3} \right) \bigg|_{x=1}^{x=\infty} = \frac{1}{3} \cdot \frac{\pi}{2} - \frac{1}{3} \cdot \tan^{-1} \left( \frac{1}{3} \right). \]

This is the same thing, which is not obvious.

In either case we know it converges.

(b)

The boxes represent

\[ \frac{1}{1^2 + q} + \frac{1}{2^2 + q} + \frac{1}{3^2 + q} + \cdots + \frac{1}{k^2 + q} \]

\[ + \int_{k+1}^{\infty} \frac{1}{x^2 + q} \, dx. \]

They are above the shaded area which explains why this is an underestimate.
The error is at most \( \frac{1}{(k+1)^2 + q} \).

If \( k = 1 \) then this is \( \frac{1}{2^2 + q} = \frac{1}{13} < 1 \).

So we can take our lower bound

\[
\frac{1}{10} + \int_{\frac{1}{2}}^{\infty} \frac{1}{x^2 + q} \, dx = \frac{1}{10} + \frac{1}{3} \cot^{-1} \left( \frac{2}{3} \right)
\]

and our upper bound

\[
\frac{1}{10} + \frac{1}{23} + \int_{\frac{1}{2}}^{\infty} \frac{1}{x^2 + q} \, dx = \frac{1}{10} + \frac{1}{23} + \frac{1}{3} \cot^{-1} \left( \frac{2}{3} \right).
\]

5. Look at \( \sum_{n=1}^{\infty} \frac{n^2 - 1}{n^4 + 1} \).

We have \( \frac{n^2 - 1}{n^4 + 1} < \frac{n^2}{n^4} \), so

\[
\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^4 + 1} < \sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

This is a p-series with \( p = 2 \), so since it converges, our original series converges too by the comparison test.

Our upper bound for \( \sum_{n=1}^{\infty} \frac{1}{n^2} \):

Upper bound for \( \sum_{n=1}^{\infty} \frac{1}{n^2} \): 

Our upper bound (with \( k = 0 \)) is

\[
\frac{1}{1^2} + \int_{1}^{\infty} \frac{1}{x^2} \, dx = 1 + \left[ \frac{-1}{x} \right]_{1}^{\infty} = 1 + \left( 0 - (-1) \right) = 2 \text{ by the integral test}.
\]
7. \( \sum_{n=1}^{\infty} (-1)^n \frac{n}{\ln \ln n} \).

This fails the \( n \)-th term test, and hence diverges,
(i.e., the \( n \)-th term test shows it diverges)

because \( \lim_{x \to \infty} \frac{x}{\ln(x)} = \lim_{x \to \infty} \frac{\frac{d}{dx} \ln(x)}{\frac{d}{dx} (\ln x)} \) (by L'Hôpital)

\[ = \lim_{x \to \infty} \frac{1}{x} = \lim_{x \to \infty} \frac{1}{x} = \infty. \]

Note this does not satisfy condition (2) for the alternating series test.