Convergence Tests for Math 142 — Frank Thorne (thorne@math.sc.edu)

This sheet will be provided to you on the exam.

The convergence tests below concern infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Sometimes we write $f(n)$ instead of a_n.

Guidelines: 1. We wrote down a series starting at a_1, but in fact the starting value isn't important. If we have a series $\sum_{n=0}^{\infty} a_n$ or $\sum_{n=91887}^{\infty} a_n$ or $\sum_{n=-5189234}^{\infty} a_n$ then the convergence tests work equally well.

2. Only the *eventual behavior* of the series matters. In **all** of the convergence tests below, it is permissible to pick some N and only look at the a_n with $n \geq N$.

3. These notes don't describe how to handle geometric series, but please make sure you know that. Also the p-series test is not described because it is a special case of the integral test.

1. The nth term test. If it is not true that $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. The integral test. Suppose that $f(x)$ is a positive, continuous, and decreasing function for $x \geq 1$.

Then, $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_{x=1}^{\infty} f(x)dx$ converges.

Also, if we estimate $\sum_{n=1}^{\infty} f(n)$ by

$$f(1) + f(2) + \cdots + f(k) + \int_{k+1}^{\infty} f(x)dx,$$

then the estimate is too low by somewhere between 0 and $f(k+1)$. (You should know how to draw pictures which explain why this is true.)

3. The alternating series test. Given an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots,$$

where (1) all the a_i are positive, (2) $a_i \geq a_{i+1}$ for each i, and $\lim_{n\to\infty} a_n = 0$, then the series converges.

4. The comparison test. Suppose you have two series

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} b_n,$$

where all the a_n and b_n are nonnegative.

- If $a_n \leq b_n$ for each n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.
- If $a_n \geq b_n$ for each n, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

5. The limit comparison test. Suppose you have two series

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} b_n,$$

where all the a_n and b_n are nonnegative.

If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, and equals some *positive* number other than 0 or ∞, then either both series converge or both series diverge.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, and $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

6. The absolute convergence test. If a series converges absolutely, then it converges.

7. The ratio test. Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists and equals some number ρ.

- If $\rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $\rho > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

8. The root test. Suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists and equals some number ρ.

- If $\rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $\rho > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.