$1 + 2 + 3 + 4 + \cdots$

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Introduction.

Srinivasa Ramanujan (1887-1920)

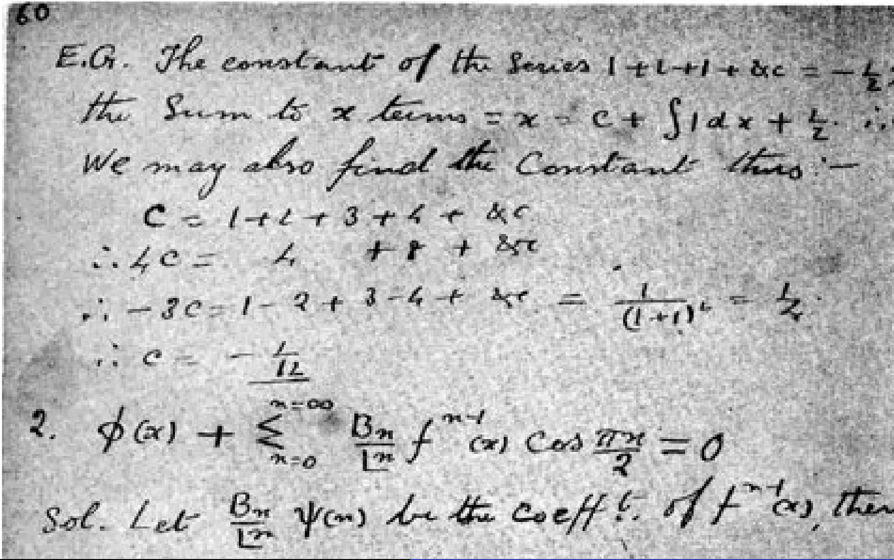
Ramanujan's second letter to Hardy

"Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfalls of divergent series. I told him that the sum of an infinite number of terms of the series: $1 + 2 + 3 + 4 + \cdots = -1/12$ *under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. I dilate on this simply to convince you that you will not be able to follow my methods of proof if I indicate the lines on which I proceed in a single letter. ..."*

(S. Ramanujan, 27 February 1913)

Warmup.

Ramanujan's proof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E.G. The constant of the series $1+1+1+\&c = -\frac{1}{2}$.

the Sum to x terms $= x = C + \int 1\,dx + \frac{1}{2}$ ∴

We may also find the Constant thus :—

$$C = 1+2+3+4+\&c$$

$$\therefore 4C = \phantom{1+{}} 4 \phantom{{}+3} + 8 + \&c$$

$$\therefore -3C = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$$

$$\therefore C = -\frac{1}{12}$$

2. $$\phi(x) + \sum_{n=0}^{n=\infty} \frac{B_n}{\underline{|n}} f^{n-1}(x) \cos\frac{\pi n}{2} = 0$$

Sol. Let $\frac{B_n}{\underline{|n}} \psi(n)$ be the coeff^t. of $f^{n-1}(x)$, then

Q.E.D.

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"The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever."

(N. Abel, 1832)

The Riemann zeta function.

Analytic continuation

Theorem (Riemann, 1859)

The zeta function has analytic continuation to all complex numbers $s \neq 1$*, with*

$$\zeta(s) = \zeta(1-s)\frac{\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}}{\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}}.$$

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Therefore,

$$\zeta(-1) = \zeta(2)\frac{\Gamma(1)\pi^{-1}}{\Gamma(-\frac{1}{2})\pi^{1/2}} = \frac{\pi^2}{6}\cdot\frac{1\times\pi^{-1}}{(-2\sqrt{\pi})\pi^{1/2}} = -\frac{1}{12}.$$

Poisson summation

The usual proof is by **Poisson summation.**

When you first see it, it looks like a piece of magic.

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Can compute $\zeta(-1) = -\frac{1}{12}$ using elementary methods?

Integration by parts.

It keeps going...

Defining $P_3(t)$ in the same manner, where

$$\int_0^1 P_3(t)dt = 0,$$

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so

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} + \frac{s}{12} - \frac{s(s+1)(s+2)}{720} - s(s+1)(s+2)\int_1^\infty \frac{P_4(t)}{t^{s+4}}.$$

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This implies that

$$\zeta(-2) = 1 + 4 + 9 + 16 + 25 + \cdots = 0,$$

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$$\zeta(-2) = 1 + 4 + 9 + 16 + 25 + \cdots = 0,$$

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and we can compute any value of $\zeta(-n)$ similarly.

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We see again that $\zeta(-1) = -\frac{1}{12}$.

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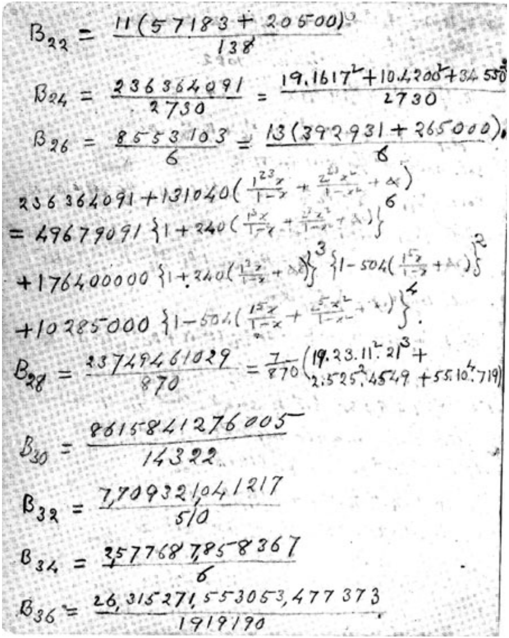
- The constant term $B_n := B_n(0)$ is called the *nth Bernoulli number.*
- If n is odd, then $B_n = 0$ (except $B_1 = -\frac{1}{2}$).



$$B_{22} = \frac{11(57183 + 20500)}{138}$$

$$B_{24} = \frac{236364091}{2730} = \frac{19.1617^2 + 104200^2 + 34550}{2730}$$

$$B_{26} = \frac{8553103}{6} = \frac{13(392931 + 265000)}{6}.$$

$$236364091 + 131040\left(\frac{1^{23}x}{1-x} + \frac{2^{23}x^2}{1-x^2} + \&c\right)$$
$$= 49679091\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^6$$
$$+ 176400000\left\{1 + 240\left(\frac{1^3x}{1-x} + \&c\right)\right\}^3\left\{1 - 504\left(\frac{1^5x}{1-x} + \&c\right)\right\}^2$$
$$+ 10285000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^4.$$

$$B_{28} = \frac{23749461029}{870} = \frac{7}{870}\left(19.23.11^2.21^3 + 2.5.25^2.4549 + 55.10^4.719\right)$$

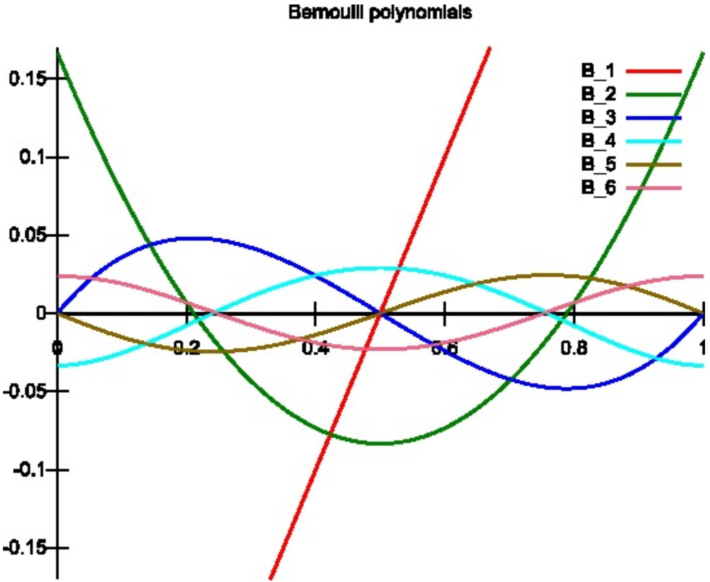
$$B_{30} = \frac{8615841276005}{14322}$$

$$B_{32} = \frac{7709321041217}{510}$$

$$B_{34} = \frac{2577687858367}{6}$$

$$B_{36} = \frac{26{,}315271{,}553053{,}477373}{1919190}$$

Some Bernoulli polynomials



The Euler-Maclaurin sum formula

Theorem

If $f \in C^\infty[0,\infty)$, then for all integers $a,\ b,\ k$ we have

$$\sum_{n=a}^{b} f(n) = \int_a^b f(t)dt + \frac{1}{2}\Big(f(a) + f(b)\Big)$$

$$+ \sum_{\ell=2}^{k} \frac{B_k}{k!}\Big(f^{(k-1)}(b) - f^{(k-1)}(a)\Big)$$

$$+ \frac{1}{k!}\int_a^b B_k(x) f^{(k)}(t)dt.$$

Example

Stirling's formula: Take $f(n) = \log(n)$:

$$\log(x!) = \sum_{n=1}^{x} \log n \longrightarrow \int_1^x \log t \;\, dt + C + \frac{1}{2} \log x.$$

Euler-Maclaurin: a special case

CHAPTER VI

Let $f(1) + f(2) + f(3) + f(4) + \cdots + f(x) = \phi(x)$ then

$$\phi(x) = c + \int f(x)\,dx + \tfrac{1}{2} f(x) + \frac{B_2}{\lfloor 2} f'(x) - \frac{B_4}{\lfloor 4} f'''(x) + \frac{B_6}{\lfloor 6} f^{v}(x) - \frac{B_8}{\lfloor 8} f^{vii}(x) + \&c$$

Sol. $\phi(x) - \phi(x-1) = f(x)$; apply V 1.

N.B. By giving any value to x, c can be found. R.S. is not a terminating series except in some special cases. Consequently no constant can be found in $\frac{1}{2} f(x) + \frac{B_2}{\lfloor 2} f'(x) - \frac{B_4}{\lfloor 4} f'''(x) + \&c$ except in those special cases. If R.S. be a terminating series, it must be some integral function of

The Ramanujan constant C_R

We have

$$\sum_{n=1}^{x} f(n) = \int_0^x f(t)dt + C_R + \frac{1}{2}f(x) + \sum_{k=2}^{\infty} \frac{B_k}{k!} f^{(k-1)}(x),$$

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where the *Ramanujan constant* C_R is defined by

$$C_R = -\frac{1}{2}f(0) - \sum_{k=2}^{\infty} \frac{B_k}{k!} f^{(k-1)}(0).$$

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The Ramanujan constant of $\sum_{n=1}^{\infty} 1$ is $-\frac{1}{2}$, because

$$\sum_{n=1}^{x} 1 = \int_0^x 1 \, dt + C_R + \frac{1}{2} \cdot 1.$$

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The Ramanujan constant of $\sum_{n=1}^{\infty} n$ is $-\frac{1}{12}$, because

$$\sum_{n=1}^{x} n = \int_0^x t \ dt + C_R + \frac{1}{2}n + \frac{1}{12}.$$

A broader definition of Ramanujan sums?

Definition

(???) We define the value of *any* infinite sum $\sum_{n=1}^{\infty} f(n)$ to be the Ramanujan constant C_R.

A convergent sum

Consider

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - 1.$$

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Can we speed up the convergence?

Warning: I am lying on this slide.

A convergent sum (cont.)

Consider instead

$$\sum_{n=1}^{\infty} \frac{1}{(n+5)^2} = \frac{\pi^2}{6} - \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}\right).$$

A convergent sum (cont.)

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Now the Ramanujan constant is

$$\begin{aligned} C_R &= -\frac{1}{2} \cdot \frac{1}{25} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot \frac{(2k)!}{5^{2k+1}} \\ &= -\frac{1}{50} + \frac{1}{750} - \frac{1}{93750} + \frac{1}{3281250} - \cdots \\ &= -0.018677028\ldots \end{aligned}$$

A convergent sum (cont.)

$\frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \cdots$ is **not** -0.018677028.

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$\frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \cdots$ is **not** -0.018677028. However,

$$\sum_{n=1}^{\infty} \frac{1}{(n+5)^2} \approx -0.018677028 + \int_0^{\infty} \frac{1}{(t+5)^2} dt = 0.181322971\ldots$$

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$$\sum_{n=1} \frac{1}{(n+5)^2} = \frac{\pi^2}{6} - \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}\right)$$
$$= 0.181322955.$$

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This calculation convinced Euler that $\zeta(2) = \frac{\pi^2}{6}$.

Rate of convergence

Question: Does the infinite series

$$C_R = -\frac{1}{2} \cdot \frac{1}{25} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot \frac{(2k)!}{5^{2k+1}}$$

converge?

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$$|B_{2n}| \sim \frac{2(2n)!}{(2\pi)^{2n}},$$

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but this can be fixed rigorously.

How to get the correct constant?

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Hardy: Introduce another parameter a.

> *"The introduction of the parameter a allows more flexibility and enables one to always obtain the "correct" constant; usually, there is a certain value of a which is more natural than other values. If $\sum f(k)$ converges, then normally we would take $a = \infty$. Although the concept of the constant of a series has been made precise, Ramanujan's concomitant theory cannot always be made rigorous."*

(B. Berndt)

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- *... and more ...*