Math 511
Practice Exam #1
Show All Work
No Calculators

1. Suppose that $A$ and $B$ are events in a sample space $S$ and that
   $P(A) = 0.4$, $P(B) = 0.3$, $P(A \cup B) = 0.5$ What is the value of:
   (a). $P(A' \cup B') =$

   Solution: First note that $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.2$
   So, $P(A' \cup B') = 1 - P(A \cap B) = 0.8$

   (b). $P(A \mid B) =$

   Solution: $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.20}{0.30} = \frac{2}{3}$

2. A box contains 10 red marbles, 10 blue marbles, 10 green marbles, 10 yellow marbles
   and 10 white marbles. Suppose that 8 marbles are chosen at random and without
   replacement. The what is the probability that
   (a). They consist of 4 each of two different colors?

   Solution: \[
   \binom{5}{2} \binom{10}{4}^2 \binom{50}{8}
   \]

   (b). They consist of 3 of one color and 5 of a second color?

   Solution: \[
   5 \times 4 \binom{10}{3} \binom{10}{5} \binom{50}{8}
   \]

3. You have a fair coin and a biased coin. The biased coin comes up heads with
   probability 0.8 and tails with probability 0.2. You choose a coin at random and flip it.
   It comes up heads. You now flip it again, what is the probability that it is again heads?

   Solution: Let $H_1$ be the event that we get a head on the first flip of the chosen coin, and
   let $H_2$ denote the event that we get a head on the second flip of the chosen coin.
   $P(H_1) = P(H_1 \cap \text{fair}) + P(H_1 \mid \text{bias}) \cdot P(H_1 \mid \text{bias})$
   \[
   = \frac{1}{2} \cdot \frac{1}{2} + \frac{4}{5} \cdot \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{4} + \frac{2}{4} = \frac{13}{20}
   \]
Now, the probability of getting two heads in two consecutive flips of the chosen coin is
\[
P(HH) = P(HH \cap \text{Fair})P(\text{Fair}) + P(HH | \text{Biased})P(\text{Biased})
\]
\[
= \frac{1}{4} \cdot \frac{1}{2} + \left(\frac{4}{5}\right)^2 \cdot \frac{1}{2} = \frac{1}{8} + \frac{8}{25} = \frac{89}{200}
\]
So, \[P(H_2 | H_1) = \frac{P(H_1 \cap H_2)}{P(H_1)} = \frac{P(HH)}{P(H_1)} = \frac{89/200}{13/20} = \frac{89}{130}.
\]

4. Suppose that a bag contains 14 good light bulbs and 6 bad light bulbs. The bulbs are chosen one at a time without replacement and tested. What is the probability that the 4th bad bulb is the 8th bulb tested?

**Solution:** Let A and B be the events:
- A - there are 3 bad bulbs among the first 7 tested.
- B - the 8th bulb is bad.

Then we want the value of \[P(A \cap B).
\]
\[
P(A \cap B) = P(B | A)P(A) = \frac{3}{13} \cdot \frac{\binom{6}{14}}{\binom{14}{4}} \cdot \frac{\binom{3}{4}}{\binom{20}{7}}
\]

5. You have to choose between two jars of candy. Jar A contains 12 pieces of chocolate and 8 pieces of hard candy. Jar B contains 6 pieces of chocolate and 14 pieces of hard candy. You randomly choose two pieces of candy from the first jar (without replacement) and discover that they are both hard candies. What is the probability that you chose the candy from Jar A?

**Solution:** Let BH denote the event that both choices are hard candy. Let A denote the event that they were chosen from Jar A, and let B denote that they were chosen from jar B. Then,
\[
P(BH) = P(BH \cap A) + P(BH \cap B) = P(BH | A)P(A) + P(BH | B)P(B = \frac{\binom{8}{2}}{\binom{20}{2}} \cdot 1 + \frac{\binom{14}{2}}{\binom{20}{2}} \cdot \frac{1}{2}.
\]
So,
\[ P(A \mid BH) = \frac{P(A \cap BH)}{P(BH)} = \frac{P(BH \mid A)P(A)}{P(BH)} = \frac{\binom{8}{2} \cdot \frac{1}{2}}{\binom{20}{2} \cdot \frac{1}{2}} = \frac{\binom{8}{2} + \binom{14}{2}}{\binom{20}{2} \cdot \frac{1}{2}} = \frac{\binom{8}{2}}{\binom{119}{2}} = \frac{28}{119} \]

6. Suppose that 4 numbers are chosen at random from the set \{1, 2, 3, 4, 5, ..., 17\}. What is the probability that their sum is even?

Solution: \[ \frac{\binom{9}{4} + \binom{9}{2} \cdot \binom{8}{2} + \binom{8}{2}}{\binom{17}{4}} = 0.5056 \]

7. (a) Suppose that 10 fair coins are independently flipped at random. What is the probability that exactly 6 of them come up heads?

Solution: \[ \binom{10}{6} \left( \frac{1}{2} \right)^{10} \]

(b) If each of 400 boxes contains 2 red marbles and 98 blue marbles and a marble is chosen at random from each box, what is the probability that at least one red marble gets picked and what is the approximate value of this probability in terms of the constant \( e \)?

Solution: \[ 1 - \left( 1 - \frac{2}{100} \right)^{400} = 1 - \left[ \left( 1 - \frac{2}{100} \right)^{100} \right]^4 \approx 1 - \left( \frac{1}{e^2} \right)^4 = 1 - \frac{1}{e^8} \]

8. If a lottery consists of drawing 10 positive digits (1 - 9) in order (with replacement). You choose some selection of digits with repetitions allowed and if some permutation of your selection agrees with the drawing, then you win. What is the probability that some permutation of 3 3 4 4 4 6 6 6 6 6 is the winning number?

Solution: \[ \frac{14!}{9^{10}} \]
9. Suppose that the events $A$, $B$, and $C$ are all independent and that $P(A) = 0.3$, $P(B) = 0.6$, and $P(C) = 0.7$. Then what is
(a) $P(A \cap (B \cup C)) = \phantom{0}$

Solution: $P(A \cap (B \cup C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) = 0.264$

(b). The probability that at least one of these events occurs? ____

Solution: This is $1 - P(A' \cap B' \cap C') = 1 - (0.7)(0.4)(0.3) = 0.916$

10. Each person in a group of 60 people is randomly assigned an ID number between 1 and 100 (inclusive). What is the probability that at least two people have the same ID number?

Solution: First count the probability that no two choose the same number and subtract that from 1. So we get, $1 - \frac{100 \cdot 99 \cdot 98 \cdot \ldots \cdot 42 \cdot 41}{100^{60}}$