Math 575
Problem Set 8

1. Find a tree and an ordering of the vertices of that tree so that a greedy coloring with respect to that ordering will require five colors.

2. Suppose that $G$ is a graph and $A$ and $B$ are induced subgraphs of $G$ such that every vertex of $A$ is adjacent to every vertex of $B$. Then we write $G = A \oplus B$.

Show that if $G = A \oplus B$, then $\chi(G) = \chi(A) + \chi(B)$ and $\omega(G) = \omega(A) + \omega(B)$.

3. **Prove:** If $G$ is $k$-chromatic, then $G$ contains at least $k$ vertices of degree at least $k - 1$.

**Solution:** Consider any good $k$-coloring of the graph with the 'colors' 1, 2, ..., $k$.

For each color $j$, there is some vertex $v$ of that color that cannot be re-colored another color (or else we could color $G$ with $k - 1$ colors. If $v$ were not adjacent to at least one vertex of each of the other colors, then we could recolor it. Thus $v$ must have degree at least $k - 1$.

**Hint:** Show that for any $k$-coloring of $G$, there is a vertex of each color that has degree at least $k - 1$.

4. **Recall that the independence number, $\beta$, of a graph is the maximum number of vertices in an independent subset of the vertices.**

Show that for any graph on $n$ vertices,

(i). $\chi(G) \geq \frac{n}{\beta(G)}$.

(ii). $\chi(G) \leq n - \beta(G) + 1$.

**Solution:**

(i). Suppose that $G$ has chromatic number $k$ and that we have colored the vertices of $G$ with the colors 1, 2, 3, ..., $k$. Let the color classes of this coloring be $V_1, V_2, \ldots, V_k$. Then $|V_i| \leq \beta(G)$ since each color class is an independent set of vertices, for every $1 \leq i \leq k$. So,

$$n = \sum_{i=1}^{k} \beta(G) = k \beta(G) = \chi(G) \beta(G).$$

And the result follows upon division by $\beta(G)$.

(ii). Let $S$ be a maximum independent set of vertices of $G$. Color all the vertices of $S$ with the color 1. Color each of the remaining $n - \beta$
vertices a different color. We have used \( n - \beta + 1 \) colors. Since the chromatic number represents the smallest number of colors needed, we get that \( \chi(G) \leq \frac{n}{\beta(G)} \).

5. **Prove:** For any graph \( \chi(G)\chi(\overline{G}) \geq n \).

   **Solution.** We have \( \chi(\overline{G}) \geq \omega(\overline{G}) = \beta(G) \). Thus, \( \chi(G)\chi(\overline{G}) \geq \chi(G)\beta(G) \geq n \), by the previous problem (part (i)).

6. **G is** \( k \)-critical with respect to the chromatic number (and we say that \( G \) is chromatic critical) if \( \chi(G) = k \) and \( \chi(G - v) = k - 1 \) for every vertex \( v \) of \( G \). Show that if \( G \) is a \( k \)-critical graph, then \( \delta(G) \geq k - 1 \).

   **Solution.** Let \( v \) be any vertex of \( G \). Then since \( G \) is \( k \)-critical, \( \chi(G - v) = k - 1 \). So color \( G - v \) with \( k - 1 \) colors. Then since \( G \) itself cannot be colored with \( k - 1 \) colors, \( v \) must be adjacent to a vertex of each of these \( k - 1 \) colors or else we could assign \( v \) one of these colors and get a good \( k - 1 \) coloring of \( G \).

7. Suppose that \( G \) is a \( k \)-chromatic graph and \( v \) and \( u \) are vertices of \( G \) such that \( N(v) \subseteq N(u) \). Show that \( G \) is not a chromatic critical graph.

8. Suppose that \( G \) is a \( k \)-chromatic graph and that \( V(G) \) has been colored using the colors \{1, 2, 3, …, \( k \)\}. Then show that for any permutation of the colors 1, 2, …, \( k \), there is a path in \( G \) on \( k \) vertices whose vertices have the colors in that order.

9. **Prove:** If \( G \) is a connected graph with maximum degree \( \Delta \) and \( G \) is not regular, then \( \chi(G) \leq \Delta \).

   **Hint:** This is harder than the rest. As a hint, begin with the fact that if \( G \) is not regular, then there exists a vertex \( v \) such that \( \delta(v) < \Delta \). Now use the fact that every connected graph has at least two vertices that are not cut-vertices to construct a permutation that will not require more than \( \Delta \) colors for a greedy coloring.