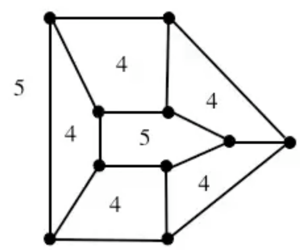
Math 575
Problem Set 13

1. Show that the square of a path P_n $(n \geq 3)$ is Hamiltonian.

2. Suppose that G is a plane graph that has 15 edges in the boundary of its exterior region and all the other regions of G contain 4, 6, or 8 edges in their boundary. Use Grinberg's Theorem to show that G cannot contain a Hamiltonian cycle.

   Solution: Grinberg's Equation reduces to $2\Delta f_4 + 4\Delta f_6 + 6\Delta f_8 = 13$, which is impossible since the left-hand expression must be even and the right hand side is odd.

3. The graph below is Hamiltonian.
   (a). Show for any Hamiltonian cycle C for the graph, that the inner 5-region lies outside C.
   (b). How many of the 4-regions are inside C, and how many are outside C?
   (c). Check your answers by looking at an actual Hamiltonian cycle in this graph.



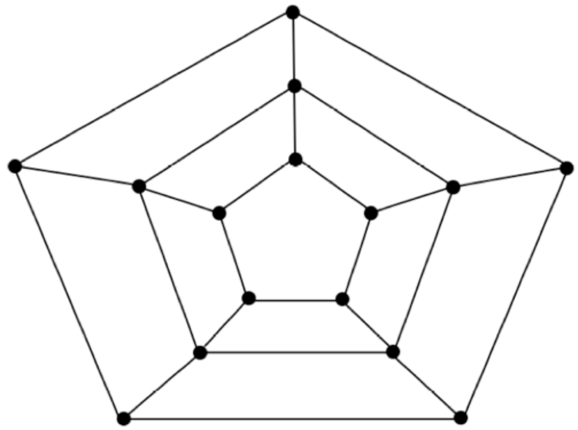
Solution:
(a). The Grinberg equation for this graph is $2\Delta f_4 + 3\Delta f_5 = 0$. If the inner 5-region lies inside the cycle C, then $\Delta f_5 = 0$, and so the Grinberg equation reduces to $2\Delta f_4 = 0$ and so we get that $\Delta f_4 = 0$.

But that means that $f_4 - f_4{}' = 0$ but this is impossible since $f_4 + f_4{}' = 5$.

(b). Since both 5-regions lie outside C, it follows that $\Delta f_5 = -2$ and so, $\Delta f_4 = 3$. Now solving the equations $f_4 + f_4{}' = 5$ and $f_4 - f_4{}' = 3$ simultaneously, we get $f_4 = 4$ and $f_4{}' = 1$. So there are four 4-regions outside C and one 4-region inside C.

(c). Do it.

4. Show that the inner 5-region of the graph below lies inside every Hamiltonian cycle.



Solution:

There are ten 4-regions and two 5-regions. So just as in the previous problem we get that $2\Delta f_4 + 3\Delta f_5 = 0$ and argue essentially the same as before only this time it follows that the inner 5-region lies outside any given Hamiltonian cycle C because there are an even number of 4-regions.

5. Suppose that G is a plane graph that has a Hamiltonian cycle C.
Suppose too, that G has
8 regions bounded by 4-cycles,
3 regions bounded by 3-cycles, (two are inside C, and one is outside C)
1 region bounded by a 5-cycle, and the exterior region is bounded by a 10-cycle.

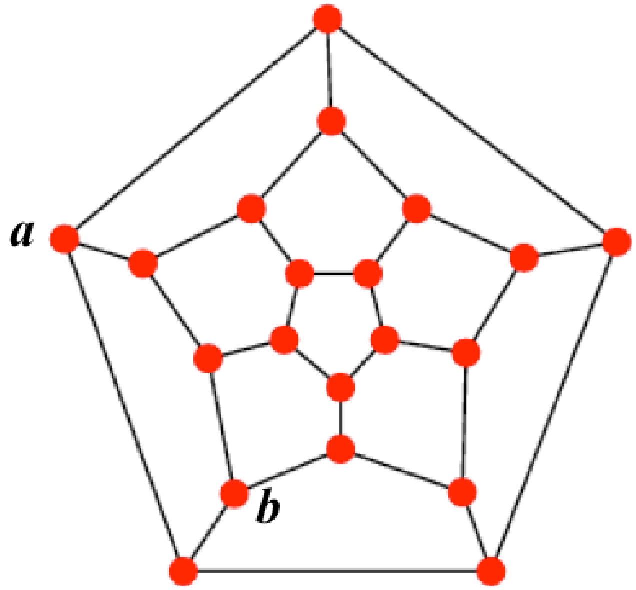
Use Grinberg's Theorem to determine how many of the regions bounded by 4-cycles lie inside C. *Explain your work carefully.*

Solution:

The Grinberg equation is $\Delta f_3 + 2\Delta f_4 + 3\Delta f_5 = 8$. Since two of the 3-regions are in C, and one is outside C, we have $\Delta f_3 = 2 - 1 = 1$. So the Grinberg equation reduces to $2\Delta f_4 + 3\Delta f_5 = 7$. Since there is just one 5-region, $\Delta f_5 = 1$ or $\Delta f_5 = -1$. So either $\Delta f_4 = 2$ or $\Delta f_4 = 5$. However, since the number of 4-regions is even, it must be that $\Delta f_4 = 2$.

So we have $f_4 + f_4{}' = 8$ and $f_4 - f_4{}' = 2$. Solving these simultaneously, we get that $f_4 = 5$ and so five of the 4-regions lie inside C.

6. (a). Show that there is no Hamiltonian path from a to b in the graph below.
(b). Show that there is no Hamiltonian cycle in the graph below with vertex b removed.



Solution:

(a). Form a new graph G^* by adding the edge ab to the given graph.

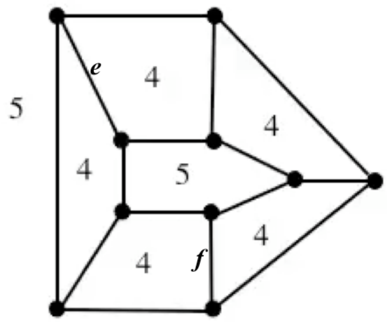
Then there is a Hamiltonian cycle in G^* that contains ab if and only if there is a Hamiltonian a-b path in the original graph.

So it is enough to show that G^* does not have a Hamiltonian cycle that contains the edge ab. Suppose then that G^* *does* have a Hamiltonian cycle C that contains the edge ab. G^* is a plane graph that has one 3-region, one 4-region, and eleven 5-regions. The Grinberg equation for G^* is $\Delta f_3 + 2\Delta f_4 + 3\Delta f_5 = 0$. But since the 4-region and 3-region both contain the edge ab in their respective boundary, it follows that $\Delta f_4 = 1$ and $\Delta f_3 = -1$ or $\Delta f_4 = -1$ and $\Delta f_3 = 1$. In the first case, we get $3\Delta f_5 = -1$ and in the second case we get $3\Delta f_5 = 1$. Both of these are impossible, and we conclude that no such cycle C can exist.

(b). Call the graph with the vertex b removed G^*. Then G^* has eight 5-regions and one 9-region. Thus the Grinberg equation is $3\Delta f_5 + 7\Delta f_9 = 0$. But $\Delta f_9 = \pm 1$, and so we get $3\Delta f_5 = \pm 7$, which is impossible.

Note that a similar argument shows that the removal of any vertex from this graph results in a non-Hamiltonian graph.

7. For the graph below, show that there is no Hamiltonian cycle that contains both of the edges *e* and *f*.



Solution: Denote the graph above by G. We could argue much as in the previous problems, but here we will demonstrate another approach. This time form a new graph from G by subdividing the edge *e* and also subdividing the edge *f*. Call the resulting graph G^*.
Now G has a Hamiltonian cycle containing *e* and *f* if and only if G^* has a Hamiltonian cycle.

However G^* has one 4-region and six 5-regions. So the Grinberg equation is $2\Delta f_4 + 3\Delta f_5 = 0$, which reduces to $3\Delta f_5 = \pm 2$ (since $\Delta f_4 = \pm 1$), and this is impossible.

8. Show that every tournament has a spanning directed path. You may argue by induction or use any result that we have discussed so far (other than this result, itself!).

Solution:
We will argue by induction noting first that the result is true for the case $n = 1$. Suppose that for some integer $n > 1$, the result holds for all tournaments on fewer than n vertices. Now let T be any tournament on n vertices.
We will be finished if we can show that T must have a spanning directed path.

Let v be any vertex of T and let A denote the in-neighborhood of v and let B denote the out-neighborhood of v. Then each of A and B has fewer than n vertices (in fact one of A or B could be empty). We will assume here that A and B are both non-empty, but it is a simple manner to argue the case where either is empty. Let P be a directed spanning path of A with initial vertex a and terminal vertex x. Let Q be a directed spanning path of B with initial vertex y and terminal vertex b. Then if we follow P from a to x, take the arc av and then the arc vy and then follow Q from y to b we have produced a spanning directed path of T from a to b.
[In the case that A is empty, just begin at v and in the case that B is empty just end at v.]