Math 575  
Problem Set 10

An orientation $\tilde{G}$ of a graph $G$ is an assignment of an orientation (direction) to each edge. We call the directed edges arcs. So if the edge $vu$ is oriented from $v$ to $u$, we refer to the resulting directed edge as the arc $(v, u)$. We denote the arcs of $\tilde{G}$ by $A(\tilde{G})$. We sometimes refer to $v$ as the tail of the arc and $u$ as the head of the arc.

Any orientation of a complete graph is referred to as a tournament.

An acyclic orientation of a graph is an orientation that does not contain any oriented cycles. A directed path (resp. directed cycle) is a path (resp. cycle) in which all the edges have the same orientation.

The indegree $\text{deg}^- (v)$ of a vertex $v$ in an oriented graph $\tilde{G}$ is the number of arcs that are oriented towards $v$; i.e., the number of arcs of the form $(u, v)$.

The outdegree $\text{deg}^+ (v)$ of a vertex $v$ in an oriented graph $\tilde{G}$ is the number of arcs that are oriented from $v$; i.e., the number of arcs of the form $(v, u)$.

1. A vertex $v$ of a directed graph is called a king if for every vertex $u$ of the graph, there is a directed path of length 1 or 2 from $v$ to $u$. So thinking of a tournament as a ‘real tournament’ where the arc $(v, u)$ indicates that $v$ beat $u$ in the competition, if $v$ is a king and $u$ is any other vertex, then either $v$ beat $u$ or $v$ beat someone who beat $u$.

Show that every tournament contains a king.

**Hint:** Choose a vertex of the tournament that has maximum outdegree.

**Solution:** Suppose that $v$ is a vertex of maximum out-degree in the tournament $T$. Then let $A$ denote the out-neighborhood of $v$ and let $B$ denote the in-neighborhood of $v$.

We claim that $v$ is a king in $T$. To verify this claim, it is enough to show that there is a directed path of length 2 from $v$ to each vertex in its in-neighborhood.

So now suppose that $u$ is any vertex in the in-neighborhood of $v$. If there is a vertex $w$ in $A$ such that $(w, u)$ is an arc of $T$, then $vuw$ is a directed path of length 2 from $v$ to $u$. On the other hand, if there is no such vertex in $A$, then it must be that $u$ dominates every vertex of $A$. Thus since $u$ dominates everything that $v$ does and $u$ also dominates $v$, the outdegree of $u$ is greater than that of $v$ - but this contradicts the choice of $v$. Thus the claim is verified and the result holds.
2. Suppose that $G$ is a graph and $\tilde{G}$ an acyclic orientation of $G$. Let $k$ denote the number of vertices in a longest directed path in $\tilde{G}$. Show that $G$ is $k$-colorable. 

**Hint:** Color a vertex $v$ with the number of vertices in a longest directed path that begins at $v$. Show that this is a good $k$-coloring.

**Solution.** Let $v$ and $u$ be adjacent vertices of $G$, and suppose that the edge $vu$ is oriented as the arc $(v, u)$. Then if the number of vertices in a longest path that originates at $u$ is $l$, the number of vertices in a longest path that originates at $v$ will be at least $l + 1$. Thus $v$ and $u$ must have different colors and the coloring is a good one.

3. Show that if $G$ is a $k$-chromatic graph, then there exists an acyclic orientation $\tilde{G}$ of $G$ in which the longest directed path contains exactly $k$ vertices.

**Hint:** Consider a coloring of $G$ with the colors 1, 2, ..., $k$. Now devise an orientation based upon these colors.

**Solution.** Simply orient each edge from the vertex with lower label the vertex with higher label. We know that there is an acyclic path in this orientation that contains at least $k$ vertices. Since the magnitudes of the vertex colors increase at we traverse a path, the path cannot contain more than $k$ vertices.