Math 575   Practice Exam 3.

1. (a). The Ramsey number \( r(n, m) \) is the smallest integer \( N \) such that every graph \( G \) on \( N \) vertices contains a \( n \)-vertex complete subgraph or a \( m \)-vertex independent set.

Solution: The smallest integer \( N \) such that every graph \( G \) on \( N \) vertices contains a \( n \)-vertex complete subgraph or a \( m \)-vertex independent set is \( r(n, m) \).

(b). If \( G \) is a graph on 36 vertices with \( \chi(G) = 5 \), then \( G \) must contain an independent set of size at least ________.

Solution: 8

(c). Draw the Grötsch graph. Solution: See your notes.

2. Suppose that \( G \) is a graph that has no induced \( K_{1,4} \) and \( \omega(G) = 4 \). Find the smallest number \( N \) such that you can verify that \( \Delta(G) \leq N \). Justify your answer.

Solution: Let \( N = 17 \). Suppose that it is not true that \( \Delta(G) \leq 17 \). Then there is a vertex \( v \) in \( G \) that has degree at least 18. Let \( H \) denote the subgraph induced by the neighborhood, \( N(v) \), of the vertex \( v \).

Then since \( r(4, 4) = 18 \), either there is an independent set of 4 vertices in \( H \) which together with \( v \) would give an induced \( K_{1,4} \) in \( G \), or there is a complete set of 4 vertices in \( H \) which together with \( v \) would give a \( K_5 \) in \( G \) – which is impossible since \( \omega(G) = 4 \).

3. Use the Chromatic Lower Bound to determine an example that shows that \( r(C_5, P_6) \geq 11 \). \textbf{Hint:} \( \chi(C_5) = 3 \). Verify that your example works.

Solution: Let \( G \) be the graph consisting of two disjoint copies of \( K_5 \). Then \( P_6 \) is not a subgraph of \( G \), and since the complement of \( G \) is \( K_{5,5} \), a bipartite graph, the complement cannot contain \( C_5 \) whose chromatic number is 3.

4. \( P_G(t) = t^5 - 5t^4 + 10t^3 - 3t^2 + 3t \) is the chromatic polynomial of some graph \( G \).

(a). What is the missing coefficient of \( t^2 \)?

Solution: Since \( P_G(1) = 0 \), the missing coefficient is 9.

(b). How many edges does \( G \) have?

Solution: 5

(c). Show that \( G \) is a bipartite graph.

Solution: \( P_G(2) = 2 > 0 \) and so it is possible to color \( G \) with two colors.
5. (a). Find the chromatic polynomial of the graph $G$ below and use it to determine the number of ways to color this graph with 3 colors.

Solution: 

$$P_G(t) = \frac{P_{C_5}(t)P_{C_4}(t)(t-2)}{t(t-1)}.$$ 

$$P_G(3) = \frac{30 \times 18}{3 \times 2} = 90.$$ So there are 90 ways to color $G$ with 3 colors.

(b). How many ways are there to color the 5-wheel $W_5$ (shown below) with 4 colors?

Solution: 

$$4 \times P_{C_5}(3) = 4 \times 30 = 120.$$ 

(c). For the graph below, list an ordering of the vertices so that a greedy coloring with respect to that ordering will require 4 colors.

Solution: $b \ c \ e \ d \ f \ a$ (among others).

6. **Prove:** every red-blue coloring of the edges of $K_{14}$ contains two disjoint red 4-cycles or two disjoint blue 4-cycles.

Solution: Suppose that we consider any red-blue coloring of the edges of $K_{14}$. Then since $r(C_4, C_4) = 6$, there must be a monochromatic 4-cycle in this coloring. WLOG we will suppose that it is a red $C_4$. Now consider the remaining 10 vertices of $K_{14}$. Again, there must be a monochromatic $C_4$ in the coloring of the edges among these 10 vertices. If it is a red $C_4$, then we are finished. So, suppose that it is a blue $C_4$. Now there are 6 remaining vertices not part of either of the monochromatic triangles we have already constructed. There is a monochromatic $C_4$ in the coloring of the edges of these 6-vertices which together with the appropriate one of the previous two $C_4$'s gives us the desired graph.
7. Show that \( r(P_4, P_4) = 5 \).

**Solution:** First we show that \( r(P_4, P_4, K_5) \leq 5 \). Consider an arbitrary red-blue coloring of the edges of \( K_5 \). We must show that there is either a red \( P_4 \) or a blue \( P_4 \). Now suppose that \( v \) is a vertex in \( K_5 \), then without loss of generality, \( v \) is incident with at least two red edges – say \( vu \) and \( vw \) are red. Let \( x \) and \( y \) be the remaining two vertices. Then either one of \( xu, xw, yu, yw \) is red and we have a red \( P_4 \) or they are all blue and we have a blue \( P_4 \).

The graph \( K_3 \cup K_1 \) shows \( r(P_4, C_4) > 4 \).

8. Prove: If \( G \) is connected but not regular, then \( \chi(G) \leq \Delta(G) \).

**Solution:** We may assume that \( G \) has at least two vertices. Since \( G \) is not regular, there is a vertex \( w \) such that \( \deg(w) < \Delta \). Now since \( G \) is connected on at least two vertices, \( G \) must have a non cut-vertex, \( v_1 \neq w \) (why?). Now let \( G_1 = G - v_1 \). Then \( G_1 \) is connected and if it is not just \( \{w\} \), then it must contain a non cut-vertex \( v_2 \neq w \). Let \( G_2 = G_1 - v_2 \) and continue this way to generate a sequence of vertices \( v_1, v_2, \ldots, v_k \) such that for each \( 1 \leq k \leq n-1 \), each \( v_k \) is adjacent to some vertex to the right of it in the sequence. So if we now color the vertices of \( G \) according to this permutation we never need to use more than \( \Delta \) colors. For suppose that we have colored the vertices \( v_1, v_2, \ldots, v_r \). Then since \( K = \{v_{r+1}, v_{r+2}, \ldots, w\} \) is connected, \( v_{r+1} \) must be adjacent to at least one vertex in \( K \) and hence there are at most \( \Delta - 1 \) colors used for the neighbors of \( v_{r+1} \) that have thus far been colored. So there must be a color free for \( v_{r+1} \). Finally, when we get to \( w \), we know that there will be a color available for \( w \) because it is adjacent to fewer than \( \Delta \) vertices of \( G \).

9. Suppose that the numbers \( \{1, 2, 3, 4, \ldots, 16\} \) are colored red, blue, and green and that we color the edges of the complete graph on \( K = \{0, 1, 2, \ldots, 21\} \) as in the proof of Schur’s Theorem.

Suppose too that the numbers 4, 11, and 19 form the vertices of a blue triangle in this coloring of the edges of \( K \). Determine a blue equation of the form \( a + b = c \).

**Solution:** \( 7 + 8 = 15 \) is a blue equation.