**The Chromatic Polynomial**

The chromatic polynomial $P_G(t)$ for a graph $G$ is the number of ways to properly color (i.e., no two adjacent vertices have the same color) the vertices of $G$ with at most $t$ colors. For a specific value of $t$, this is a number, however (as shown below) for a variable $t$, $P_G(t)$ is a polynomial in $t$ (and hence its name).

It is easy to compute $P_G(t)$ for certain special classes of graphs.

**Theorem.** (a) $P_{K_n}(t) = t(t-1)(t-2)\cdots(t-n+1)$

(b) $P_{K_1}(t) = t^n$

(c) $P_T(t) = t(t-1)^{n-1}$ for any tree $T$ on $n$ vertices.

Two recursion formulas

**Theorem.** If $a$ and $b$ are non-adjacent vertices of the graph $G$, then $P_G(t) = P_{G+ab}(t) + P_{G,ab}(t)$.

**Proof.** $P_G(t)$ is the number of ways to color $V(G)$ using at most $t$ colors.

$= \text{number of ways with } a, b \text{ the same color} + \text{number of ways with } a, b \text{ different colors}$.

But, $P_{G+ab}(t)$ is the number of ways to color $V(G)$ with $a, b$ different colors.

And $P_{G,ab}(t)$ is the number of ways to color $V(G)$ with $a, b$ the same color.

**Corollary.** For any graph $G$, $P_G(t)$ is a polynomial in the variable $t$.

**Proof.** Repeated application of the previous theorem will result in an expression for $P_G(t)$ as a sum of the form $P_G(t) = \sum_{i=1}^{n} a_i P_{n_i}(t)$ where the $a_i$'s may be 0. Since $P_{n_i}(t)$ is a polynomial in $t$, it follows that $P_G(t)$ is a polynomial in $t$.

**Example:** Here we use the notation of brackets around a graph to represent the chromatic polynomial of the graph.

$P_{C_4}(t) =$

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**Theorem.** If $a$ and $b$ are adjacent vertices of the graph $G$, then $P_G(t) = P_{G-ab}(t) - P_{G,ab}(t)$.

*Proof.* Apply the previous theorem to the graph $H = G - ab$. Then since $H_{ab} = G_{ab}$ and $H = G - ab$, we get, $P_{G-ab}(t) = P_H(t) = P_{H+ab}(t) + P_{H,ab}(t) = P_G(t) + P_{G,ab}(t)$. Thus, $P_{G-ab}(t) = P_G(t) + P_{G,ab}(t) = P_G(t) - P_{G,ab}(t)$.

**Theorem.** Suppose that $G$ has components $A_1, A_2, \ldots, A_k$. Then $P_G(t) = P_{A_1}(t)P_{A_2}(t)\cdots P_{A_k}(t)$.

*Proof.* This is obvious.

**Theorem.** For any graph $G$ on $n$ vertices, $P_G(t)$ is a polynomial in $t$ having degree $n$, and also the following statements hold.

(a). It’s leading term is $t^n$.

(b). The constant term of $P_G(t)$ is 0.

(c). The coefficient of $t^{n-1}$ is $m$ – the number of edges of $G$.

(d). The coefficients of $P_G(t)$ alternate in sign.

(e). If the coefficient of $t$ is not zero, then $G$ is connected.

(f). If $G$ has any edges, then the sum of the coefficients of $P_G(t)$ is 0.

*Proof.*

Parts (a) – (d) follow by a common induction argument on $m$, the number of edges of $G$.

(e). If $G$ has more than one component, then it would follow from (b) and the previous theorem that the constant term of $P_G(t)$ would be 0.

(f). If $G$ has any edges, then $V(G)$ cannot be properly colored with one color and so

$$P_G(1) = 0.$$ However, $P_G(1)$ is equal to the sum of the coefficients of $P_G(t)$.

**Theorem.** For any cycle $C_n$, $P_{C_n}(t) = (t-1)^n + (-1)^n(t-1)$.

*Proof.* This is a simple induction using the second recursion formula,

$$P_G(t) = P_{G-ab}(t) - P_{G,ab}(t).$$

**Theorem.** Suppose that $S$ is a complete subset of $k$ vertices of the connected graph $G$. Suppose that $G - S$ is disconnected with components $A$ and $B$.

Let $H$ be the subgraph induced by $A \cup S$ and let $M$ be the subgraph induced by $B \cup S$.

Then $P_G(t) = \frac{P_H(t)P_M(t)}{t(t-1)\cdots(t-k+1)}$.

*Proof.* There are $P_H(t)$ ways to color the vertices of $H$ with at most $t$ colors and then after that $P_M(t)$ ways to color the vertices of $M$ [and hence complete the coloring of $V(G)$] using the same $t$ colors. So the total number of ways to color the vertices of $G$ using a set of $t$ colors is $P_G(t) = \frac{P_H(t)P_M(t)}{t(t-1)\cdots(t-k+1)}$. □

The special case of this where $S$ consists of a single vertex makes it easy to reduce the computation of the chromatic polynomial for a graph having a cut-vertex.
Corollary. Suppose that \( G \) is a connected graph that has a cut vertex \( v \) and suppose that \( G - v \) has \( r \) components \( A_1, A_2, \ldots, A_r \). For each \( i = 1, 2, \ldots, r \) let \( H_i \) denote the subgraph induced by \( A_i \cup \{v\} \). Then \( P_G(t) = \frac{P_{A_1}(t)P_{A_2}(t)\cdots P_{A_r}(t)}{t^{r-1}} \).

Proof. There are \( P_{H_1}(t) \) ways to properly color \( H_1 \) and then \( \frac{P_{H_i}(t)}{t} \) ways to properly color each of the \( H_i \)'s with \( i > 1 \).

Note that if \( a < \chi(G) \), then \( P(a) = 0 \).

Some curiosities about the Chromatic Polynomials.

1. Tutte and some of his students observed that for any triangulated planar graph \( G \), \( P_G(t) \) has a root close to \( \frac{3+\sqrt{5}}{2} \).

2. Let \( \phi = \frac{1+\sqrt{5}}{2} \). Then for any triangulated planar graph \( G \), \( P_G(\phi \sqrt{5}) > 0 \).

3. Let \( A \) be the vertex-adjacency matrix of the graph \( G \) with \( V(G) = \{v_1, v_2, \ldots, v_n\} \), i.e., \( A \) is the \( n \times n \) matrix with \( a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \).

Theorem (Wilf). If \( \lambda_{\text{max}} \) denotes the largest eigenvalue of \( A \), then \( \chi(G) \leq 1 + \lambda_{\text{max}} \).

Exercises:

1. Let \( G \) be the graph that consists of an \( n \)-cycle with an additional vertex adjacent to all the vertices of the cycle [such a graph is called an \( n \)-wheel [or just wheel if \( n \) is understood], and is denoted by \( W_n \). Determine the chromatic polynomial for \( W_n \).

Solution: \( P_{W_n}(t) = tP_{C_n}(t-1) = t(t-2)^n + (-1)^n t(t-2) \).

2. Let \( P \) be the chromatic number of an even cycle, what is the value of \( P(2) \)?

Solution: \( P(2) = 2 \) since there are 2 ways to color an even cycle with two colors.

3. In how many ways can the vertices of an 8-cycle be properly colored using the colors red, blue, and green?

Solution: Since \( P_{C_8}(t) = (t-1)^8 + (t-1) \), there are \( P_{C_8}(3) = 2^8 + 2 = 66 \) ways.
4. What is the chromatic polynomial for the graphs below?

Solution: \[ P(t) = \frac{P_{C_1}(t)P_{C_4}(t)}{t(t-1)} = \frac{(t-1)^{10} + (t-1)(t-1)^4 + (t-1)}{t(t-1)}. \]

Solution: \[ P(t) = (t-1)^{10} + (t-1)^7. \]

Solution: \[ P_G(t) = t(t-1)(t-2)^3(t-3). \]

5. Prove: If the chromatic polynomial of \( G \) is \( P_G(t) = t(t-1)^{n-1} \), then \( G \) is a tree on \( n \) vertices.

Solution: Since \( P_G(t) = t(t-1)^{n-1} = t^n - (n-1)t^{n-1} + \cdots + 1 \), we can see that \( G \) is a connected graph on \( n \) vertices that has \( n-1 \) edges and hence must be a tree.

6. How many ways are there to color a cycle on 6 vertices using red, blue, and green if each color must be used at least once?

Solution: The number of ways to color a cycle on 6 vertices using red, blue, and green if each color must be used at least once is 60.

7. Explain why you would have a proof of the 4-color theorem if you could show that \( P_G(4) > 0 \) for any planar graph \( G \).

Solution: If \( P_G(4) > 0 \), then there is at least one way to properly color there vertices of \( G \) with 4 colors.

8. Is \( P(t) = t^4 - 3t^3 + 3t^2 \) the chromatic polynomial of some graph \( G \)?

Solution: No.