1. The *independence number*, $\alpha$, is the cardinality of a largest independent set of vertices of $G$.

Show that if $G$ is any graph, then

(a) $\chi(G) \leq n - \alpha(G) + 1$  
(b) $\chi(G)\alpha(G) \geq n$

**Hints:**

(a). Consider a maximum independent set $A$.

(b). Consider that a coloring of $G$ with $k = \chi(G)$ colors yields a partition of $V(G)$ into $k$ independent sets $\{V_1, V_2, V_3, \ldots, V_k\}$.

**Solution:** (a). Choose a maximum independent set $S$ and color all its vertices with the same color, and then color all the remaining $n - \alpha(G)$ vertices a different color and we get a $n - \alpha(G) + 1$ coloring of $G$.

(b). Let $\chi(G) = k$ and suppose that $\{S_1, S_2, \ldots, S_k\}$ is a partition of the vertex set of $G$ into $k$ independent sets.

Thus $|S_i| \leq \alpha$ for each $i$ and so, $n = \sum_{i=1}^{k} |S_i| \leq \sum_{i=1}^{k} \alpha = k\alpha$, and hence $k \geq \frac{n}{\alpha}$.

2. Suppose that $G$ has chromatic number $k$ and for some vertices $v$ and $u$, $N(v) \subseteq N(u)$, then show that $G$ is not $k$-critical.

**Solution:** Suppose that $G - v$ could be colored with $k - 1$ colors. Then take that coloring and assign the vertex $u$ the same color as that given to $v$, and we get a $k - 1$ coloring of $G$ -- contrary to $\chi(G) = k$.

3. (a). Show that the Grotsch graph has chromatic number 4.

(b). In general, why must the graph $G_k$, as constructed in class, be triangle free and have chromatic number $k$?

(c). How many vertices are in the graph $G_5$?

4. Show that if $G$ is $k$-chromatic, then for any tree $T$ on $k$ vertices, $G$ has a subgraph isomorphic to $T$.

**Solution:** Since $G$ is $k$-chromatic, it must contain a subgraph $H$ that is $k$-critical. But then every vertex of $H$ has degree at least $k - 1$ and so $H$ must contain every tree on $k$ vertices.

5. A *greedy coloring* of a graph $G$ is obtained by taking a permutation of the vertices of $G$ and then coloring each vertex in turn with one of the numbers 1,2,\ldots,$n$ assigning each vertex in turn the smallest color not used by any of its neighbors.

(a). Find a tree and a greedy coloring of the tree that requires 5 colors.
**Solution:** The tree below requires 4 colors if the vertices are colored in alphabetical order. By appropriately connecting two copies of this tree, you can get a tree and a greedy coloring that requires 5 colors.

![Tree Diagram]

(b). What can you say about the chromatic number of a tree on \( n \geq 2 \) vertices?

**Solution:** It is 2.

6. A *digraph* \( \tilde{G} \) (sometimes called an *oriented graph*) is an orientation of a simple graph \( G \). So each edge \( e = ab \) becomes either \( (a, b) \) [if the edge is oriented from \( a \) towards \( b \)] or \( (b, a) \) [if the edge is oriented from \( b \) towards \( a \)].

The set of ordered pairs that represent the oriented edges is called the set of *arcs* of \( \tilde{G} \) and is denoted by \( A(\tilde{G}) \).

A *tournament* is an oriented complete graph.

An *oriented path* in \( \tilde{G} \) is a sequence \( v_1, v_2, v_3, \ldots, v_m \) in which \( (v_i, v_j) \) is an arc of \( \tilde{G} \) for each \( i = 1, 2, \ldots, m-1 \). If \( (v_m, v_1) \) is also an arc, then we say that \( v_1, v_2, v_3, \ldots, v_m \) is an oriented cycle.

\( \tilde{G} \) is *acyclic* if it has no oriented cycles.

**Prove:** Every tournament contains an oriented spanning path; i.e., an oriented path that contains all the vertices of \( \tilde{G} \).

**Hint:** Argue by induction. Keep in mind that if \( \tilde{G} \) is a tournament, then so is \( \tilde{G} - v \) for any vertex \( v \).

7. How many distinct tournaments are there on \( n \) vertices?

**Hint:** You must make a decision about each edge in \( K_n \).
8. Let $G$ be a graph and let $\tilde{G}$ be an acyclic orientation of $G$. Suppose that the longest oriented path in $G$ has length $k$, (i.e., contains $k + 1$ vertices). Then show that $\tilde{G}$ is $k$-colorable.

**Hint:** Color a vertex $v$ with $r$ where $r$ is the length of a longest oriented path that starts at $v$.

9. Let $\chi(G) = k$. Show that every acyclic orientation of $G$ contains an oriented path on $k + 1$ vertices. *Note that this was not the statement on the handout in class.*

**Hint:** Consider a partition of $G$ into $k$ independent sets.