Math 547
Problem Set #2

1. Let $G$ be a group and $a$ an element of $G$. Let $cl(a)$ denote the conjugacy class of $a$ and $C(a)$ the centralizer of $a$.
Show that the mapping $\gamma : cl(a) \to C(a)$ defined by $\gamma(b) = xC(a)$ where $b = xax^{-1}$ is well defined.
i.e., You must show that if $b$ is conjugate to $a$ in more than one way,
say $b = xax^{-1}$ and $b = yay^{-1}$, then $xC(a) = yC(a)$.

2. Show that there are exactly two groups of order $p^2$ for any prime $p$ and describe these two groups explicitly.
Hint: Recall that (letting $Z$ denote the center of a group $G$):
if $G / Z$ is cyclic, then $G$ is Abelian.

3. It follows from the class equation that every $p$-group has a non-trivial center.
Give an example of a nontrivial group whose center is trivial.

4. Show that every non-Abelian group of order 8 has exactly 2 elements in its center.
What is the center of the Quaternions $\mathbb{Q}_8$?

5. What is the maximum order of an element of $\mathbb{Z}_n \times \mathbb{Z}_m \times \mathbb{Z}_k$?