Math 547  
Problem Set #1

1. List all the partitions of 7.
Solution: There are 15 such partitions.
7, 6+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2, 3+2+1+1, 3+1+1+1+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1.

2. (a). List all (up to isomorphism) of the Abelian groups of order 64.
(b). List all (up to isomorphism) of the Abelian groups of order 288.
Solution: (a). We get one Abelian group of order 64 for each partition of 6 since $2^6 = 64$. The eleven partitions of 6 are:
6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, and 1+1+1+1+1+1.
So the eleven Abelian groups of order 64 are:
$\mathbb{Z}_{64}$, $\mathbb{Z}_{32} \times \mathbb{Z}_2$, $\mathbb{Z}_{16} \times \mathbb{Z}_4$, $\mathbb{Z}_{16} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_8 \times \mathbb{Z}_8$, ..., $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b). Since $288 = 32 \times 9 = 2^5 \times 3^2$ there are $7 \times 2 = 14$ such groups.
For example, $\mathbb{Z}_{32} \times \mathbb{Z}_9$, $\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

3. Prove: If $G$ is an Abelian group of order 15, then $G$ is cyclic.
Solution: Since 3 and 5 are relatively prime, $\mathbb{Z}_3 \times \mathbb{Z}_5 \equiv \mathbb{Z}_{15}$.

4. Let $A$ and $B$ be normal subgroups of the group $G$ such that $A \cap B = \{e\}$.
We showed in Math 546 that $AB$ is a group. The group $AB$ is called the internal direct product of $A$ and $B$.

(i). Verify that $ab = ba$ for all $a \in A$, $b \in B$.

(ii). Show that $A \times B$ is isomorphic to $AB$.

5. Suppose that $G$ is a group and we define the relation $a \sim b \iff xa = bx$ for some element $x$ of $G$. If $a \sim b$ we say that $a$ is conjugate to $b$.
Verify that $\sim$ is an equivalence relation on $G$.
The conjugacy class of $a \in G$ is $cl(a) = \{b : b \text{ is conjugate to } a\}$.
i.e., $cl(a)$ is the equivalence class of $a$ under $\sim$.
Let $G$ be any group having center $Z = Z(G)$. Show that for any $a \in G$,
$cl(a) = \{a\} \iff a \in Z$. 
From your text:
Section 11 (Page 110) : #1, 3, 4, 5, 7, 15, 16