Math 547
Problem Set #1

1. List all the partitions of 7.
   Solution: There are 15 such partitions.
   7, 6+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2, 3+2+1+1, 3+1+1+1+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1.

2. (a). List all (up to isomorphism) of the Abelian groups of order 64.
   (b). List all (up to isomorphism) of the Abelian groups of order 288.
   Solution: (a). We get one Abelian group of order 64 for each partition of 6 since $2^6 = 64$. The eleven partitions of 6 are:
   6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, and 1+1+1+1+1+1.

   So the eleven Abelian groups of order 64 are:
   $Z_{64},\ Z_{32} \times Z_2,\ Z_{16} \times Z_4,\ Z_{16} \times Z_2 \times Z_2,\ Z_8 \times Z_8, \cdots,\ Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$

   (b). Since $288 = 32 \times 9 = 2^5 \times 3^2$ there are $7 \times 2 = 14$ such groups.
   For example, $Z_{32} \times Z_9$, $Z_8 \times Z_4 \times Z_3 \times Z_3$, and $Z_4 \times Z_4 \times Z_2 \times Z_3 \times Z_3$.

3. **Prove**: If *G* is an Abelian group of order 15, then *G* is cyclic.
   Solution: Since 3 and 5 are relatively prime, $Z_3 \times Z_5 \cong Z_{15}$.

4. Let *A* and *B* be normal subgroups of the group *G* such that $A \cap B = \{e\}$.
   We showed in Math 546 that *AB* is a group. The group *AB* is called the *internal direct product* of *A* and *B*.

   (i). Verify that $ab = ba$ for all $a \in A,\ b \in B$.

   (ii). Show that $A \times B$ is isomorphic to AB.

5. Suppose that *G* is a group and we define the relation $a \sim b \Leftrightarrow xa = bx$ for some element *x* of *G*. If $a \sim b$ we say that *a* is *conjugate* to *b*.

   Verify that $\sim$ is an equivalence relation on *G*.

   The *conjugacy class* of $a \in G$ is $cl(a) = \{b : b \text{ is conjgate to } a\}$.
   i.e., $cl(a)$ is the equivalence class of a under $\sim$.

   Let *G* be any group having center $Z = Z(G)$. Show that for any $a \in G$, $cl(a) = \{a\} \Leftrightarrow a \in Z$.

From your text:
Section 11 (Page 110) : #1, 3, 4, 5, 7, 15, 16