Hints and Solutions

Recall that
*If $E$ is a finite extension of $F$ and $K$ is a finite extension of $E$, then $[K:F] = [K:E][E:F]$. 

23. **Hint**: Use (*) above and the fact that $F \subseteq F(\alpha) \subseteq E$.

24. **Solution**: It is enough to show that $x^2 - 2$ has no zeros in $\mathbb{Q}(\sqrt{2})$. For this it is enough to show that $\sqrt{2} \not\in \mathbb{Q}(\sqrt{2})$. However, if $\sqrt{2} \in \mathbb{Q}(\sqrt{2})$, then $2 = \deg(\sqrt{2}, \mathbb{Q})$ would divide $3 = \deg(\sqrt{2}, \mathbb{Q})$.

26. **Hint**: Let $\alpha \in D$, $\alpha \neq 0$. It is enough to show that $\alpha^{-1} \in D$.
   Since $E$ is a finite extension of $F$, $E$ is algebraic over $F$ and so since $\alpha \in E$,
   $F(\alpha) = \{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n : a_i \in F\}$, where $n = \deg(\alpha, F)$.

29. **Hint**: Suppose that $\alpha \in E$ is a zero of $p(x)$. Then $\deg(p(x)) = [F(\alpha):F]$ - now use (*).

30. **Hint**: Since $F(\alpha)$ is a finite extension of $F$, it is an algebraic extension and so $\alpha^2 \in F(\alpha)$ must be algebraic over $F$. Suppose that $F(\alpha^2) \neq F(\alpha)$ and consider the value of $[F(\alpha):F(\alpha^2)]$.

31. You may use 31.11 in your text for this.