Math 547 – Practice Exam #3

1. (a). Explicitly describe the elements of the field \( \mathbb{Q}(\pi) \).
(b). Explicitly describe the elements of the field \( \mathbb{Q}(\sqrt{2}) \).
(c). Give a basis for the field \( \mathbb{Q}(\sqrt{3}, \sqrt{2}, i) \) over \( \mathbb{Q} \).
(d). Define algebraic extension.

2. Prove: If \( F \subseteq K \subseteq E \) are fields and \( K \) is a finite extension of \( F \) and \( E \) is a finite extension of \( K \), then \( [E:F] = [E:K][K:F] \).

3. Suppose that \( \gamma \) is a zero of \( p(x) = x^2 + 2x + 3 \in \mathbb{Z}_5[x] \) in some extension field \( E \).
   Note: \( p(x) = x^2 + 2x + 3 \) is irreducible in \( \mathbb{Z}_5[x] \); you need not verify this.
   (a). How many elements are there in \( \mathbb{Z}_5(\gamma) \)? Explain.
   (b). Express the product \( (1 + 2\gamma)(3 + \gamma) \) in the form \( a + b\gamma, a, b \in \mathbb{Z}_5 \).
   (c). Find an expression (in terms of \( \gamma \)) for the other zero of \( p(x) = x^2 + 2x + 3 \) in \( E \).

4. Let \( D \) be an integral domain with \( F \subseteq D \subseteq E \) where \( F \) and \( E \) are fields and \( E \) is a finite extension of \( F \). Show that \( D \) is a field.

5. Show directly that \( \alpha = \sqrt{i + \sqrt{3}} \) is an algebraic number and determine its degree. Fully justify your answer.
   Hint: You may take as given that \( \sqrt{i + \sqrt{3}} \notin \mathbb{Q}(i, \sqrt{3}) \).

6. Given that \( \pi \) is transcendental, show that \( \sqrt{\pi} \) cannot be algebraic of degree at most 2.

7. Suppose that \( p(x) \in F[x] \) is irreducible of degree \( n \) and that \( \alpha \) is a zero of \( p(x) \) in some extension field \( E \). Thus \( p(x) \) is the minimal polynomial for \( \alpha \).
   Let \( S = \{1, \alpha, \alpha^2, \ldots, \alpha^n\} \). Show that \( S \) is linearly independent in \( F(\alpha) \).
   Note: Argue directly, you may not use the fact that \( S \) is a basis for \( F(\alpha) \).

8. Let \( \alpha \) be algebraic in \( E \) over \( F \) and suppose that \( p(x) \) is its minimal polynomial.
   Then show that if \( f(x) \in F[x] \) with \( f(\alpha) = 0 \), then \( p(x) \) divides \( f(x) \).