Math 547–Review Problems–Exam 1
Note that problem 8 had the word ‘commutative’ inserted to make it a little nicer.

Be able to do problems such as those below and the problems assigned on problem sets or the following problems from your textbook.

**Textbook Problems:**
Page 110–111: 1, 2, 3, 4, 7, 9, 13 14(c), 15, 18, 20.
**Solutions:**
#4: The order of (2,3) is \(\text{lcm}(3, 5) = 15\).

#14: \(\text{lcm}(3, 4) = 12\).

#18: No. When written as a direct product of prime-power cyclic groups the two representations are not identical.

#20: Yes – both are isomorphic to \(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_9\).

Page 174–176: 1, 5, 8, 11, 13, 14, 15, 29, 30, 47, 50, 55.
**Solutions:**
#8: Not a ring (this set consists of just the positive integers and it is not a group under addition.)

#14: Just 1 and \(-1\).

#30: *A unit* in a ring \(R\) with identity is an element \(x\) for which there is a multiplicative inverse \(y\); i.e., there exists a \(y\) such that \(xy = yx = 1\).

#50: Just show it is closed under subtraction and multiplication.

Page 182–184: 1, 2, 5, 9, 11, 14, 15, 16, 29.
**Solutions:**
#2: \(x = 3\).

#14: \[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
-2 & -2 \\
1 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

#16: \(n\) must be the *smallest* such positive integer.

Page 326–327: 1, 2, 3, 5, 7, 9, 13.
**Solution:** #2: 27


1. The *center* of a ring \(R\) is the set \(Z(R) = \{a : ar = ra \text{ for all } r \text{ in } R\}\).
   Show that the center of a ring is a subring of the ring.
   **Solution:** Just show that \(Z(R)\) is closed under subtraction and multiplication.
2. What are the units of the polynomial ring \( R[x] \) where \( R \) is the set of real numbers?  
**Solution:** The units are \( p(x) = k, \; k \neq 0 \).

3. Find an example of a ring and two elements \( a \) and \( b \) such that \( ab = 0 \) but \( ba \neq 0 \).  
**Solution:** \( A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{Z}) \) works since \( AB = 0 \) but \( BA \neq 0 \).

4. Suppose that \( R \) is a ring with identity and \( a \) is an element of \( R \) such that \( a^2 = a \). Is \( S = \{ara : r \in R \} \) a subring of \( R \)?  
**Solution:** Yes.

5. Is \( S = \left\{ \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} : a, b \in \mathbb{Z} \right\} \) a subring of \( M_2(\mathbb{Z}) \)?  
**Solution:** No. It is not closed under multiplication.  
For example, \( \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 11 \\ 9 & 13 \end{bmatrix} \notin S \).

6. Is \( S = \left\{ \begin{bmatrix} a & a^2 \\ b & b \end{bmatrix} : a, b \in \mathbb{Z} \right\} \) a subring of \( M_2(\mathbb{Z}) \)?  
**Solution:** Yes.

7. Show that if \( A \) and \( B \) are ideals of a ring \( R \), then \( A + B = \{a + b : a \in A, \; b \in B \} \) is an ideal of \( R \).  
**Solution:** Let \( x, y \in A + B \). Then \( x = a_1 + b_1, \; y = a_2 + b_2 \) for some \( a_1, a_2 \in A, \; b_1, b_2 \in B \).  
Now, \( x - y = (a_1 + b_1) - (a_2 - b_2) = (a_1 - a_2) + (b_1 - b_2) \in A + B \) and so \( A + B \) is closed under subtraction. Also for any element \( r \) in \( R \), \( rx = r(a_1 + b_1) = ra_1 + rb_1 \in A + B \) since \( ra_1 \in A, \; rb_1 \in B \) because \( A \) and \( B \) are ideals. Similarly \( xr \in A + B \) and so \( A + B \) absorbs multiplication.

8. Suppose that an ideal \( A \) of a ring \( R \) contains a unit, then show that \( A = R \).  
**Solution:** Suppose that \( u \) is a unit in \( A \) (note that \( R \) must be a ring with identity if we can talk about units). Then since \( u \) is a unit, there exists a \( w \) in \( R \) such that \( uw = 1 \). Hence \( 1 = uw \in A \) since \( A \) absorbs multiplication. Now, let \( r \) be any element of \( R \), then \( r = r \cdot 1 \in A \) since \( A \) absorbs multiplication.
9. What are the possible ideals of a field?
   Solution: Just the field itself and \{0\}. This is a consequence of exercise 4.

10. Show that every group of order 35 is cyclic.
    Solution: The number of 5-Sylow subgroups divides 35 and is of the form 5n+1. The only such number is 1. Similarly, the number of 7-Sylow subgroups is 1. Thus G has exactly one subgroup of order 5 and exactly one subgroup of order 7. Those groups can only account for \(5 + 7 - 1 - 11\) elements of G. Hence any other element of G must have order 35.

11. What are the units of \(\mathbb{Z}_{24}\)?
    Solution: The units of \(\mathbb{Z}_{24}\) are 1, 5, 7, 11, 13, 19, 23 the elements of \(\mathbb{Z}_{24}\) that are relatively prime to 24.

12. (a). If \(R\) is a commutative ring of characteristic 3, then simplify the expansion of \((a+b)^3\).
    Solution: \((a+b)^3 = a^3 + b^3\).

   (b). If \(R\) is a commutative ring of characteristic 4, then simplify the expansion of \((a+b)^5\).
    Solution: \((a+b)^5 = a^5 + ab^4 + 2a^2b^3 + 2a^3b^2 + a^4b + b^5\).