Math 547 2006 – Practice Exam 1

1. Define:
   (a). a is a nilpotent element of a ring R.
   Solution: An element a is nilpotent if $a^n = 0$ for some integer n.

   (b). b is a *unit* in the ring R.
   Solution: A unit in a ring with identity is an element u that has a multiplicative inverse; i.e., there exists a w such that $uw = wu = 1$.

   (c). p-Sylow subgroup.
   Solution: If p is a prime divisor of the order of the group G, then H is a p-Sylow subgroup of G if the order of H is p^k and p^{k+1} does not divide the order of G.

   (d). Describe how the field of quotients Q_D is formed from an integral domain D.

   [See your notes]

2. **Prove**: The center of a p-group is non-trivial.
   Proof. Let G be a p-group for the prime p. Let S be a complete set of representatives for the equivalence relation of conjugacy in G. Let A be the set of all those elements of S that do not belong to the center of G. Then the class equation for G is

   [See your notes]

   And so now, [finish the proof]

   [See your notes]

3. (a). Show that the conjugacy relation on a group G is an equivalence relation.

   [See your notes]

   (b). Use the Sylow theorems to show that every group of order 99 has a normal subgroup of order 9.
   Solution: Since the only number of the form $3k+1$ that divides 99 is 1, there is a unique 3-Sylow subgroup of G. Thus, since gHg^{-1} is also a subgroup of order 9 (and hence also a 3-Sylow subgroup), it must be that $gHg^{-1} = H$ for every $g \in G$ Thus H is normal..

4. (a) **Prove**: Every group of order 77 is cyclic.

Solution: Let G have order 77. By work similar to that in #3, there is a unique subgroup of order 7 and a unique subgroup of order 11. But this can account for only 7 + 11 – 1 – 17 elements. Thus the remaining 60 elements of G must have order 77 and are generators for G.

(b). Let R be a commutative ring with identity and let A be an ideal of R. Suppose that A contains a unit. Show that $A = R$.

Solution: Suppose that u is a unit in A. Let w be such that $uw = 1$. Then since A absorbs multiplication, it follows that $1 = uw$ is in A. Now let x be any element of R. Again, using the fact that A absorbs multiplication, $x = 1 \cdot x \in A$. Thus $R = A$.

5. (a). How many (up to isomorphism) Abelian groups are there of order $2^5 3^3$?
Solution: There are 7 partitions of 5 and 3 partitions of 3 and hence there are $7 \times 3 = 21$ Abelian groups of this order.

(b). What is the order of the element (3, 4) in $Z_{15} \times Z_{20}$?
Solution: $lcm(5,5) = 5$.

(c). H is an Abelian group of order 24 that has an element of order 12 and two elements of order 2. Determine an external direct product of prime-power cyclic groups that is isomorphic to H.
Solution:
An Abelian group of order 24 is one of $Z_8 \times Z_3$, $Z_4 \times Z_2 \times Z_3$, or $Z_2 \times Z_2 \times Z_2 \times Z_3$.
But then the other conditions of the problem indicate that H must be $Z_4 \times Z_2 \times Z_3$.

(d). What are the units of the ring $M_2(R)$ where R is the set of real numbers?
Solution: The units are the invertible matrices; i.e., those with a non-zero determinant.

(e). What is the characteristic of the ring $Z_3 \times Z_6$?
Solution: 6

6. (a). Suppose that R is an integral domain and a is an idempotent in R. Show that $a = 0$ or $a = 1$.

Solution: $a^2 = a \Rightarrow a^2 - a = 0 \Rightarrow a(a-1) = 0$. So since R has not zero-divisors, either $a = 0$ or a–1 = 0. So $a = 0$ or a =1.

(b). Give an example of a commutative ring with identity that is not an integral domain.
Solution: Z_6 since $2 \times 3 = 0$

7. Is $S = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \text{ are real numbers} \right\}$

(a). A subring of $M_2(Z)$?

Solution: Yes. Check that it is closed under subtraction and multiplication.

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} c & d \\ d & c \end{bmatrix} = \begin{bmatrix} ac+bd & bc+ad \\ bc+ad & ac+bd \end{bmatrix}$$

and $\begin{bmatrix} a & b \\ b & a \end{bmatrix} - \begin{bmatrix} c & d \\ d & c \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ b-d & a-c \end{bmatrix}$

(b). An ideal of $M_2(Z)$?

Solution:

No, it does not absorb multiplication, e.g., $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \notin S$

Justify your answers.