1. (a). List all the partitions of \{a, b, c\} – there are five of them.

(b). How many partitions of \{a, b, c, d\} are there?
   **Hint**: How many are there with just one part? With two Parts? Etc.

The next two exercises show that there is a natural one-to-one correspondence between the equivalence relations on a set \(S\) and the partitions of \(S\).

2. Let \(P = \{A_1, A_2, \ldots, A_k\}\) be a partition of the set \(S\). Then note that the relation 
   \(a \sim b \iff a, b \in A_i \) for some \(1 \leq i \leq k\) is an equivalence relation.

(a). What property of partitions justifies the claim that \(\sim\) is reflexive?
(b). What property of partitions justifies the claim that \(\sim\) is symmetric?
(c). What property of partitions is used to justify the claim that \(\sim\) is transitive?

One way to show that two sets \(A\) and \(B\) are equal is to show that each is a subset of the other. In other words, if \(A \subseteq B\) and \(B \subseteq A\), then \(A = B\). We use this idea in the next exercise.

3. **Theorem.** Suppose that \(\sim\) is an equivalence relation on the set \(S\).
   Then for \(x, y \in S\), \([x] \cap [y] \neq \emptyset \Rightarrow [x] = [y]\).

   **Proof.** Suppose that \(\sim\) is an equivalence relation on the set \(S\), and that \(x, y \in S\), with \([x] \cap [y] \neq \emptyset\). We must show that \([x] = [y]\). We will do this by showing that \([x] \subseteq [y]\) and \([y] \subseteq [x]\).

   Since \([x] \cap [y] \neq \emptyset\), there must be some element \(a \in [x] \cap [y]\).

   Now suppose that \(z \in [x]\)…. 

   **Finish the argument** by showing that \(z\) must also belong to \([y]\).

4. Show that every integer can be written in the form \(7a + 3b\) for some choice of integers \(a\) and \(b\). Example: \(10 = 7 + 3\), \(10 = -2 \times 7 + 4 \times 6\).
   **Hint**: What is the greatest common divisor of 3 and 7?

   Let \((S,*)\) be a binary system. Then an element \(e\) of \(S\) which has the property that for every element \(a\) of \(S\) \(e * a = a * e = a\), is called an identity element for \((S,*)\).

   Example: \((\mathbb{Z}, +)\) has the identity element 0 and \((\mathbb{Z}, \times)\) has the identity element 1.
5. For each of the binary systems below, determine which have an identity element and identify the identity element.

a. \((P(\{a, b, c, d\}), \cap)\).

b. \((\mathbb{Z}^+, \oplus)\) where \(a \oplus b = \frac{ab}{a + b}\).

c. \((\mathbb{R}, \odot)\) where \(a \odot b = a \ln b\).

d. \((\mathbb{Z}^+, +)\).

e. \(\{i, x, y, z, r, s\}, \circ\) where the operation is defined by the table:

\[
\begin{array}{cccccc}
& i & x & y & z & r & s \\
i & i & x & y & z & r & s \\
x & x & i & s & r & z & y \\
y & y & r & i & s & x & z \\
z & z & s & r & i & y & x \\
r & r & y & z & x & s & i \\
s & s & z & x & y & i & r \\
\end{array}
\]

f. The set \(M_2\) of all \(2 \times 2\) matrices where the operation is matrix multiplication.

g. \(\{1, 2, 3, 4, 5, 6, 7, 8\}, \Delta\) where \(a \Delta b = \min\{a, b\}\).

h. \((\mathbb{Z}^+, \Delta)\) where \(a \Delta b = \min\{a, b\}\).

i. \((\mathbb{Z}, \#)\) where \(a \# b = a + b - 2ab\).

6. For each of the systems in Problem 5 that have an identity determine if each element of the system has an inverse.

7. A binary system \((S, \ast)\) is a group if it is a semigroup that has an identity and each element of \(S\) has an inverse. Which of the systems in problem 5 are groups?