Math 546
Problem Set 7

1. Suppose that \( f: \mathbb{Z}^+ \to (0,1) \) and that
\[
\begin{align*}
    f(1) &= 0.841137..., \\
    f(2) &= 0.275434..., \\
    f(3) &= 0.216779..., \\
    f(4) &= 0.211490..., \\
    f(5) &= 0.122832..., \\
    f(6) &= 0.201508...
\end{align*}
\]
Let \( x \) denote the number constructed in the proof that no function from \( \mathbb{Z}^+ \) to \((0,1)\) can be onto. Determine the value of \( x \) to 6 decimal places.

Solution:
\( x = 0.987549... \)

2. Prove: If \( f:S \to T \) is an isomorphism of the binary structure \((S,\ast)\) onto the binary structure \((T,\#)\), then \( f^{-1}:T \to S \) is an isomorphism of the binary structure \((T,\#)\) onto the binary structure \((S,\ast)\). You may assume as already known that \( f^{-1} \) is a bijection. Note that by definition, \( f^{-1}(x) = a \iff f(a) = x \).

Solution: : Here’s how you might phrase your argument.
Proof. Let \( x \) and \( y \) be any elements of \( T \). Then we must show that
\[
f^{-1}(x \# y) = f^{-1}(x) \# f^{-1}(y).
\]
However, since \( f \) is onto we know that there exist \( a \) and \( b \) in \( S \) such that \( f(a) = x \), \( f(b) = y \). Hence, \( f^{-1}(x) = a \), \( f^{-1}(y) = b \), and so
\[
f^{-1}(x \# y) = f^{-1}(f(a) \# f(b)) = f^{-1}(f(a \ast b)) = a \ast b = f^{-1}(x) \# f^{-1}(y).
\]
Which was what we wanted.

3. Suppose that \( f: R \to R \) defined by \( f(x) = 2x + 1 \) is an isomorphism from the binary structure \((R,\ast)\) to \((R,\times)\) (where the \( \times \) refers to ordinary multiplication). Determine the operation \( \ast \).
Solution: Because \( f \) is an isomorphism, \( f(a \ast b) = f(a) \times f(b) \).
Hence, \( 2(a \ast b) + 1 = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 \), and solving this for \( a \ast b \) gives, \( a \ast b = a + b + 2ab \).