Math 546 — Problem Set #6

1. Determine if the two binary structures defined below are isomorphic. If they are, then provide an isomorphism. If they are not, then explain how you know that.

\[
\begin{array}{ccc}
\ast & a & b \\
a & b & c \\
b & c & a \\
c & a & b \\
\end{array}
\quad
\begin{array}{ccc}
\# & x & y \\
x & x & y \\
y & y & z \\
z & z & x \\
\end{array}
\]

Solution: The function \( a \rightarrow z, b \rightarrow y, c \rightarrow x \) is an isomorphism.

2. Determine if the two binary structures defined below are isomorphic. If they are, then provide an isomorphism. If they are not, then explain how you know that.

\[
\begin{array}{ccc}
\ast & a & b \\
a & a & b \\
b & c & b \\
c & b & a \\
\end{array}
\quad
\begin{array}{ccc}
\# & x & y \\
x & x & y \\
y & z & z \\
z & y & z \\
\end{array}
\]

Solution: These are not isomorphic. Why? The left structure has two idempotents and the right structure has just one.

3. The two binary structures below are isomorphic, what must the isomorphism be?

\[
\begin{array}{cccccc}
\ast & a & b & c & d & e \\
a & a & c & b & c & b \\
b & b & c & a & c & d \\
c & d & c & c & d & e \\
d & d & a & d & c & e \\
e & c & c & d & c & e \\
\end{array}
\quad
\begin{array}{cccccc}
\# & x & y & z & u & v \\
x & x & z & u & z & y \\
y & y & y & u & v & y \\
z & z & v & u & x & u \\
u & v & y & y & u & z \\
v & v & y & x & v & y \\
\end{array}
\]

Solution: \( a \rightarrow x, b \rightarrow z, c \rightarrow u, d \rightarrow v, e \rightarrow y \).

Just note the number of times each element appears in each table.

4. Show that \((R, \cdot)\) is isomorphic to \((R, +)\) by exhibiting a specific isomorphism. Note that we are talking about ordinary addition and multiplication here.

**Hint:** This is a common function that you've used a lot.

**Solution:** \( f(x) = \ln x \) works here.
5. Is \((\mathbb{Z}, +)\) isomorphic to \((\mathbb{Z}^+, +)\)? Justify your answer.
   Solution: No. \((\mathbb{Z}, +)\) contains the idempotent 0, but \((\mathbb{Z}^+, +)\) does not have an
   idempotent element.

6. If \((S, \ast)\) is a binary system, then an element \(e\) in \(S\) is said to be an identity element
   if \(a \ast e = a\) and \(a \ast e = a\) for every element \(a\) in \(S\).
   Suppose that \((S, \ast)\) is a binary system that contains an identity element. Show that
   if \((A, \#)\) is isomorphic to \((S, \ast)\), then \((A, \#)\) also contains an identity element.
   Solution: Suppose that \(f : S \rightarrow A\) is an isomorphism. Let \(e\) be the identity
   element for \((S, \ast)\), and let \(w = f(e)\). We will show that \(w\) is an identity element
   for \((A, \#)\).

   Now let \(a\) be an arbitrary element of \(A\). We must show that \(w \# a = a \# w = a\).

   Since \(f\) is onto, there is some \(x\) in \(S\) such that \(a = f(x)\).

   Hence, \(w \# a = f(e) \# f(a) = f(e \ast a) = f(a) = a\). Similarly, \(a \# w = a\) and we are
   finished.

7. Determine a bijection \(f : \mathbb{Z} \rightarrow \mathbb{Z}^+\).
   Solution: One choice (of many) would be: \(f(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2n + 1 & \text{if } n \leq 0 \end{cases}\)

   In Your Text: Page 34 #3, 5, 7, 9, 11.