1. Prove: If $G$ is Abelian, then every subgroup of $G$ is normal.

**Solution:** We noted this in class today.

Proof. If $H$ is a subgroup of the Abelian group $G$ and $g \in G$, $h \in H$, then $ghg^{-1} = hgg^{-1} = he = h \in H$.

2. Prove: If $H$ is a subgroup of $G$, then for any $g$ in $G$, $gHg^{-1}$ is also a subgroup of $G$.

**Solution:** Note that $gHg^{-1} = \{ ghg^{-1} : h \in H \}$

Clearly the identity is $e = geg^{-1} \in gHg^{-1}$.

If $x$ and $y$ belong to $gHg^{-1}$, then $x = gh_{1}g^{-1}$, $y = gh_{2}g^{-1}$ for some elements $h_{1}, h_{2} \in H$. Thus, $xy = gh_{1}g^{-1}gh_{2}g^{-1} = gh_{1}h_{2}g^{-1} \in gHg^{-1}$. So $gHg^{-1}$ is closed.

Finally if $x = ghg^{-1} \in H$, then $x^{-1} = (ghg^{-1})^{-1} = gh^{-1}g^{-1} \in H$ (since $h^{-1} \in H$).

3. Prove the theorem below.

**Theorem.** Let $G$ be a group and $H$ a subgroup of $G$. Then the following are equivalent:

(i). $H \triangleleft G$.

(ii). For every $g$ in $G$, $gH = Hg$.

(iii). For every $g$ in $G$, $gHg^{-1} = H$.

**Solution:** (i) $\Rightarrow$ (ii). Suppose that $H \triangleleft G$ and let $x$ be any element of $gH$.

Then $x = gh$ for some $h$ in $H$. Thus, $h_{1} = ghg^{-1} \in H$. And so,

$x = gh = ghg^{-1}g = h_{1}g \in Hg$. Thus we have shown that $gH \subseteq Hg$, Essentially the same argument shows that $Hg \subseteq gH$ and hence $gH = Hg$.

Now show (ii) implies (iii) and (iii) implies (i).

4. (a). If $H$ is a subgroup of the group $G$ and $[G : H] = 2$, then $H \triangleleft G$.

**Hint:** Consider problem 3(ii).

**Solution:** Suppose that $g$ is any element of $G$. If $g$ is in $H$, then $gH = H = Hg$.

If $g$ does not belong to $H$, then $gH$ is the left coset that is different from $H$ and $Hg$ is the right coset that is different from $H$ and so $gH = Hg$.

(b). Show that $A_{4} \triangleleft S_{4}$.
5. $A_4$ has exactly one subgroup of order 4, namely
$K = \{i, \ (1, 2)(3, 4), \ (1, 3)(2, 4), \ (1, 4)(1, 3)\}$.
Show that $K$ is a normal subgroup of $A_4$.

**Hint:** Refer to problems 2 & 3.

**Solution:** Since $K$ is the only subgroup of order 4 and since $gKg^{-1}$ is a subgroup of order 4, then $gKg^{-1} = K$ and so $K$ is normal by problem 3.

6. Recall that $SL(2, R) = \{A \in GL(2, R) : \det(A) = 1\}$. Show that $SL(2, R) \triangleleft GL(2, R)$.

**Hint:** If $A$ and $B$ are both $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.