1. How many generators of the cyclic group \( \mathbb{Z}_n \) are there?

   Is this the same as the number of generators of any other cyclic group of order \( n \)?

   **Solution:** A number in \( \mathbb{Z}_n \) generates \( \mathbb{Z}_n \) if and only if it is relatively prime to \( n \).

   Hence, since there are \( \phi(n) \) integers in \( \mathbb{Z}_n \) that are relatively prime to \( n \), there are \( \phi(n) \) generators for \( \mathbb{Z}_n \).

2. Suppose that \( a \) is an element of the group \( G \) of order \( n \). Suppose that \( a^k = e \) and that \( k < n \). Is it necessarily true that \( k \) divides \( n \)?

   **Solution:** No. Suppose that \( a \) is an element of order 5 in a group of order 20 (for example, the number 4 in \( \mathbb{Z}_{20} \)). Then \( a^{15} = e \) and 15 < 20, but 15 does not divide 20.

3. Show that if \( H \) and \( K \) are subgroups of a group \( G \), then \( H \cap K \) is a subgroup as well.

   **Solution:** We need to show that \( H \cap K \) contains the identity, is closed under the group operation and that each element in \( H \cap K \) has its inverse also in \( H \cap K \).

   Suppose that \( x, y \in H \cap K \). Then both \( x \) and \( y \) belong to \( H \) and both belong to \( K \) as well. Since they are both in \( H \) and \( H \) is closed, \( xy \) is in \( H \), and similarly since they are both in \( K \), \( xy \) is in \( K \). Thus \( xy \in H \cap K \) and so \( H \cap K \) is closed.

   Now suppose that \( x \) is any element of \( H \cap K \). Then since \( x \) is in \( H \) and \( H \) is a subgroup, \( x^{-1} \in H \). Similarly, \( x^{-1} \in K \) and so \( x^{-1} \in H \cap K \).

4. Show that every cyclic group is Abelian.

   **Solution:** Suppose that \( G \) is a cyclic group that is generated by the element \( g \).

   Let \( x \) and \( y \) be arbitrary elements of \( G \). we must show that \( xy = yx \).

   Since \( G \) is generated by \( g \), there must exist integers \( r \) and \( s \) such that \( x = g^r \), \( y = g^s \). But then \( xy = g^r g^s = g^{r+s} = g^s g^r = yx \).

5. Note that \( \phi(5) = \phi(8) = \phi(10) = \phi(12) = 4 \). Are the groups \( U(5), U(8), U(10), U(12) \) isomorphic to one another?

   **Solution:** No. \( U(8) \) and \( U(12) \) are both isomorphic to the Klein 4-group (and hence to each other), while \( U(5) \) and \( U(10) \) are both cyclic.

6. Let \( G \) be a group and \( Z = \{a \in G : ag = ga \text{ for all } g \in G \} \). Show that \( Z \) is a subgroup of \( G \).

   **Solution:** We need to show that \( Z \) contains the identity, is closed under the group operation and that each element in \( Z \) has its inverse also in \( Z \).
Since $ex = xe = x$ for any $x$ in $G$, $e$ satisfies the condition to belong to $Z$. If $a$ and $b$ belong to $Z$, then $(ab)x = a(bx) = a(xb) = (ax)b = (xa)b = x(ab)$ for any $x$ in $G$ and so $ab$ satisfies the condition to belong to $Z$. Finally if $a$ is in $Z$, then $a^{-1}x = (x^{-1}a)^{-1} = (ax^{-1})^{-1} = xa^{-1}$ and so $a^{-1}$ satisfies the condition to belong to $Z$. Note that we used the property $(xy)^{-1} = y^{-1}x^{-1}$ here as well as the fact that $(x^{-1})^{-1} = x$.

**Note:** The group $Z$ is called the Center of $G$.

7. If every proper subgroup of a group $G$ is cyclic, then must $G$ itself be cyclic?  
**Solution:** No, the Kline 4-group is a counter-example.

8. Let $G$ be a group of order $n$ and let $k$ be relatively prime to $n$.  
Show that the function $\gamma : G \to G$ defined by $\gamma(x) = x^k$ is a bijection.  
**Solution:** There is a quarter riding on this one.